Score: 9

MATH 170 HW#4

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Exercise 3.16

(a) Since the first entry of the row vector $(-\mathbf{c}'_\mathbf{B}\mathbf{B}^{-1}\mathbf{b}, -\mathbf{c}'_\mathbf{B}\mathbf{B}^{-1})$ is the negative of the current cost, it is always weakly increasing. We will show that the second part of the row vector, $-\mathbf{c}'_\mathbf{B}\mathbf{B}^{-1}$, is strictly increasing lexicographically. Let $\overline{\mathbf{B}}$ be the new basis matrix after the iteration and $\mathbf{c}'_{\overline{\mathbf{B}}}$ be the corresponding cost vector.

$$-\mathbf{c}'_{\overline{\mathbf{B}}}\overline{\mathbf{B}}^{-1} = -\begin{bmatrix} c_1 & \dots & c_{l-1} & c_j & c_{l+1} & \dots & c_m \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & -\frac{u_1}{u_l} & 0 & \dots & 0 \\ 0 & 1 & \dots & -\frac{u_2}{u_l} & 0 & \dots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & \dots & \frac{1}{u_l} & 0 & \dots & 0 \\ 0 & \dots & \dots & -\frac{u_{l+1}}{u_l} & 1 & \dots & \vdots \\ \vdots & & & \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{u_l} & 0 & \dots & 1 \end{bmatrix} \mathbf{B}^{-1}$$

$$= -\begin{bmatrix} c_1 & \dots & c_{l-1} & -\frac{1}{u_l}\sum_{i=1,i\neq l}^m c_i u_i + \frac{c_j}{u_l} & c_{l+1} & \dots & c_m \end{bmatrix} \mathbf{B}^{-1}$$

$$= -\mathbf{c}'_\mathbf{B}\mathbf{B}^{-1} + \begin{bmatrix} 0 & \dots & 0 & \frac{1}{u_l}\sum_{i=1,i\neq l}^m c_i u_i - \frac{c_j}{u_l} + c_l & 0 & 0 & 0 \end{bmatrix} \mathbf{B}^{-1}$$

$$= -\mathbf{c}'_\mathbf{B}\mathbf{B}^{-1} + \left(\frac{1}{u_l}\sum_{i=1,i\neq l}^m c_i u_i - \frac{c_j}{u_l} + c_l\right)(l \text{ th row of } \mathbf{B}^{-1})$$

Note that the coefficient is equal to $\frac{1}{u_l} * (-\bar{c}_j)$, which is positive. Since $\mathbf{B}$ is lexicographically positive, its rows are lexicographically positive. Therefore, $-\mathbf{c}'_\mathbf{B}\mathbf{B}^{-1}$ is strictly increasing lexicographically each iteration, and the row vector $(-\mathbf{c}'_\mathbf{B}\mathbf{B}^{-1}\mathbf{b}, -\mathbf{c}'_\mathbf{B}\mathbf{B}^{-1})$ is strictly increasing lexicographically.

(b) $\mathbf{B}^{-1}$ is obtained by adding the multiples of the pivot rows to each rows. For $i \neq l$, we will add $-\frac{u_i}{u_l} * (l \text{ th row})$ to the i th row. If $u_i \leq 0$, then clearly we are adding a lexicographically nonnegative row vector to the row and hence it remains lexicographically positive. If $u_i > 0$, then by our pivot rule, $\frac{1}{u_l} * (l \text{ th row}) \overset{L}{<} \frac{1}{u_i} * (i \text{ th row}) \Leftrightarrow (i \text{ th row}) - \frac{u_i}{u_l} * (l \text{ th row}) \overset{L}{>} 0$. Again, the rows remain lexicographically positive. For the l th row, since

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we obtain it by dividing the original row by u_l, which is positive, it is still lexicographically positive. Therefore, $\mathbf{B}^{-1}$ remains lexicographically positive through out the algorithm.

(c) Since the row vector $(-\mathbf{c}_\mathbf{B}'\mathbf{B}^{-1}\mathbf{b}, -\mathbf{c}_\mathbf{B}'\mathbf{B}^{-1})$ is strictly increasing lexicographically and it has a 1-1 correspondence with the basis matrix, the simplex method will not cycle and terminate in finite steps.

Exercise 3.17 Auxiliary problem:

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$$
\begin{array}{ll}
\text{minimize} & x_6 + x_7 + x_8 \\
\text{subject to} & \left\{ \begin{array}{rrrrrrrrl}
x_1 & +3x_2 & & +4x_4 & +x_5 & +x_6 & & & = 2 \\
x_1 & +2x_2 & & -3x_4 & +x_5 & & +x_7 & & = 2 \\
-x_1 & -4x_2 & +3x_3 & & & & & +x_8 & = 1
\end{array} \right. \\
& x_1, \ldots, x_8 \geq 0
\end{array}
$$

Phase I: Start with a bfs $(0, 0, 0, 0, 0, 2, 2, 1)$.

	-5	-1	-1	-3	-1	-2	0	0	0
x_6	2	1	3	0	4	1	1	0	0
x_7	2	1	2	0	-3	1	0	1	0
x_8	1	-1	-4	3	0	0	0	0	1

Choose x_3 to enter the basis and x_8 exit.

	-4	-2	-5	0	-1	-2	0	0	1
x_6	2	1	3	0	4	1	1	0	0
x_7	2	1	2	0	-3	1	0	1	0
x_3	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	1	0	0	0	0	$\frac{1}{3}$

Choose x_1 to enter the basis and x_6 exit.

	0	0	1	0	7	0	2	0	1
x_1	2	1	3	0	4	1	1	0	0
x_7	0	0	-1	0	-7	0	-1	1	0
x_3	1	0	$-\frac{1}{3}$	1	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$

Now drive x_7 out of the basis and let x_2 in.

	0	0	0	0	0	0	1	1	1
x_1	2	1	0	0	-17	1	-2	3	0
x_2	0	0	1	0	7	0	1	-1	0
x_3	1	0	0	1	$\frac{11}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$

We finally obtain a bfs for the original LP and its associated tableau. We can go to Phase II:

$$\begin{array}{c|c|ccccc}
 & -7 & 0 & 0 & 0 & 3 & -5 \\ \hline
x_1 & 2 & 1 & 0 & 0 & -17 & 1 \\
x_2 & 0 & 0 & 1 & 0 & 7 & 0 \\
x_3 & 1 & 0 & 0 & 1 & \frac{11}{3} & \frac{1}{3}
\end{array}$$

Choose x_5 to enter the basis and x_1 to exit.

$$\begin{array}{c|c|ccccc}
 & 3 & 5 & 0 & 0 & -82 & 0 \\ \hline
x_5 & 2 & 1 & 0 & 0 & -17 & 1 \\
x_2 & 0 & 0 & 1 & 0 & 7 & 0 \\
x_3 & \frac{1}{3} & -\frac{1}{3} & 0 & 1 & \frac{28}{3} & 0
\end{array}$$

Choose x_4 to enter the basis and x_2 to exit.

$$\begin{array}{c|c|ccccc}
 & 3 & 5 & \frac{82}{7} & 0 & 0 & 0 \\ \hline
x_5 & 2 & 1 & \frac{17}{7} & 0 & 0 & 17 \\
x_4 & 0 & 0 & \frac{1}{7} & 0 & 1 & 0 \\
x_3 & \frac{1}{3} & -\frac{1}{3} & -\frac{4}{3} & 1 & \frac{28}{3} & 0
\end{array}$$

The optimal cost is 3 and the corresponding optimal solution is $(0, 0, \frac{1}{3}, 0, 2)$.

Exercise 3.19

(a) Since $\bar{c}_2$ is negative, the current bfs must be degenerate so as to be optimal. Also, δ must be positive and satisfies $\delta + \frac{2}{3}\gamma = 0$ to ensure the optimality after the change of basis. We conclude that $\alpha = 1, \beta = 0, \gamma = -3, \delta = 2, \eta = 2$ is a possible choice of parameter values.

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(b) If $\delta, \alpha, \gamma < 0$, then the optimal cost will be $-\infty$. We conclude that $\alpha = -1, \beta = 1, \gamma = -1, \delta = -1, \eta = -1$ is a possible choice of parameter values.

(c) If $\beta, \delta > 0$, then the current solution is feasible but not optimal. We conclude that $\alpha = 1, \beta = 1, \gamma = 1, \delta = 2, \eta = 1$ is a possible choice of parameter values.

Exercise 3.20

(a) We just need to ensure the current solution is feasible. Thus, the ranges of values are: $\alpha, \gamma, \delta, \eta, \xi, \in \mathbb{R}$, $\beta \geq 0$.

(b) If $\beta < 0$, $\alpha \geq 0$, then the current solution is infeasible, and all feasible direction cannot drive x_2 to be positive. Thus, the ranges of values are: $\gamma, \delta, \eta, \xi, \in \mathbb{R}$, $\beta < 0, \alpha \geq 0$.

(c) If $\beta \geq 0$, then the current solution is feasible. If at least one of δ, γ, ξ is negative, then the current basis is not optimal. Thus, the ranges of values are: $\gamma, \delta, \xi, \in \mathbb{R}$ with at least one of them negative, $\beta \geq 0, \alpha, \eta \in \mathbb{R}$.

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(d) If $\beta \geq 0$, then the current solution is feasible. Analyzing all the possibilities, we have found that the only way to ensure a $-\infty$ optimal cost after one iteration is to let $\gamma < 0$, $\delta, \xi \geq 0$. If the third row is the pivot row, then we can never make a negative column. In other words, we need the second row to be the pivot row, which means $\eta > \frac{4}{3}$. Only the fourth column is possible to be negative, and we need $\alpha < 0$, $\delta + 2\frac{\gamma}{\eta} < 0$ to guarantee that. Thus, the ranges of values are: $\gamma, \alpha < 0$, $\delta, \xi, \beta \geq 0$, $\eta > \frac{4}{3}$, $\delta + \frac{2\gamma}{\eta} < 0$.

(e) If $\beta \geq 0$, then the current solution is feasible. x_6 is a candidate for entering the basis if $\gamma < 0$. If x_3 leaves the basis when x_6 entering, then $\frac{2}{\eta} < \frac{3}{2} \Leftrightarrow \eta > \frac{4}{3}$. Thus, the ranges of values: $\alpha, \delta, \xi, \in \mathbb{R}$, $\beta \geq 0$, $\gamma < 0$, $\eta > \frac{4}{3}$.

(f) x_7 is a candidate for entering the basis if $\xi < 0$. Since the solution and objective value remain unchanged after x_7 entering, the current solution is degenerate. That said, $\beta = 0$. Thus, the ranges of values are: $\alpha, \delta, \gamma, \eta \in \mathbb{R}$, $\beta = 0$, $\xi < 0$.

Exercise 3.22

(a) If $b = 0$, clearly it is feasible. Assume $b > 0$. Consider the auxiliary problem:

$$\begin{array}{ll} \text{minimize} & y \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} a_i x_i + y = b \\ & x_1, \ldots, x_n, y \geq 0 \end{array}$$

The original LP is infeasible iff $y_{min} \neq 0$. The Phase I tableau is

	$-b$	$-a_1$	$-a_2$	$\ldots$	$-a_n$	0
y	b	a_1	a_2	$\ldots$	a_n	1

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From the tableau, we can see that $y = b$ is the optimal solution iff $a_i \leq 0$ for all $i = 1, \ldots, n$. Therefore, we conclude that our criteria for feasibility is as follows:

- If $b = 0$, then the LP is feasible.
- If $b > 0 \,(< 0)$, then the LP is feasible iff $\exists i \in \{1, \ldots, n\}$ such that $a_i > 0 \,(< 0$ respectively$)$.

(b) Since the optimal cost is finite, there is a bfs that is an optimal solution. The bfs $\mathbf{x}$ has the form $x_i = 0 \; \forall i \neq j, x_j = \frac{b}{a_j}$ for some $j \in \{1, \ldots, n\}$. Therefore, we would like to choose $j = \arg\min_j \left\{ \frac{c_j}{a_j} | a_j \neq 0, \, j = 1, \ldots, n \right\}$. For j with $a_j = 0$, if $c_j < 0$, then we can have $-\infty$ cost which contradicts our assumption. If $c_j \geq 0$, then we will always make $a_j = 0$ so we can ignore it. We conclude that our method that chooses x_j with $j = \arg\min_j \left\{ \frac{c_j}{a_j} | a_j \neq 0, \, j = 1, \ldots, n \right\}$ as our basic variable will yield the optimal solution.

Exercise 3.26 Let our original LP and the big-M auxiliary problem be as follows

$$
\begin{array}{llcll}
\text{minimize} & \mathbf{c}'\mathbf{x} & \qquad & \text{minimize} & \mathbf{c}'\mathbf{x} + M\sum_{i=1}^{m} y_i \\
\text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} & & \text{subject to} & \mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{b} \\
& \mathbf{x} \geq 0 & & & \mathbf{x}, \mathbf{y} \geq 0
\end{array}
$$

(a) Assume otherwise $\mathbf{x}$ is not an optimal solution to the original LP. Then there exists an $\mathbf{x}^* \in \mathbb{R}^n$ in the feasible set of the original LP such that $\mathbf{c}'\mathbf{x}^* < \mathbf{c}'\mathbf{x}$. Note that $(\mathbf{x}^*, \mathbf{y})$ is a feasible solution to the big-M problem because $\mathbf{y} = \mathbf{0}$. Thus, we have $\mathbf{c}'\mathbf{x}^* + M\sum_{i=1}^{m} y_i \geq \mathbf{c}'\mathbf{x} + M\sum_{i=1}^{m} y_i \Leftrightarrow \mathbf{c}'\mathbf{x}^* \geq \mathbf{c}'\mathbf{x}$. Contradiction. We conclude that $\mathbf{x}$ is an optimal solution to the original LP.

(b) Assume otherwise the original LP is feasible. Then there exists an $\mathbf{x}^* \in \mathbb{R}^n$ that is a bfs to the original LP. Consider the $\mathbb{R}^{n+m}$ vector $(\mathbf{x}^*, \mathbf{0})$. It is clearly a bfs to the big-M problem with cost $\mathbf{c}'\mathbf{x}^*$. Note that $\mathbf{c}'\mathbf{x}^* < \mathbf{c}'\mathbf{x} + M\sum_{i=1}^{m} y_i$ because $\mathbf{y} \neq 0$ and M is big enough. This implies that the simplex method would not terminate with $(\mathbf{x}, \mathbf{y})$. Contradiction. We conclude that the original LP is infeasible.

(c) When the simplex method terminated, it has discovered a feasible direction $\mathbf{d} = (\mathbf{d_x}, \mathbf{d_y})$ such that $\mathbf{d} \geq \mathbf{0}$ with at least one entry positive and $\mathbf{c}'\mathbf{d_x} + M\sum_{i=1}^{m} d_{y_i} < 0$. We will show that $\mathbf{d_y} = \mathbf{0}$. Assume otherwise there is a $i \in \{1, \ldots, m\}$ with $y_i > 0$. Then $\mathbf{c}'\mathbf{d_x} + M\sum_{i=1}^{m} d_{y_i} \geq \mathbf{c}'\mathbf{d_x} + My_i > 0$ as M is big enough. Contradiction. Thus, $\mathbf{d_y} = \mathbf{0}$. We will now proof our main result by contradiction again. Assume otherwise the original LP is feasible and has finite optimal cost. Let $(\mathbf{x}^*, \mathbf{y}^*)$ be the optimal solution to the original LP and $(\mathbf{x}, \mathbf{y})$ be the bfs associated with $\mathbf{d}$. Since $\mathbf{d_y} = 0$, we have $\mathbf{A}(\mathbf{x} + \mathbf{d_x}) + \mathbf{y} = \mathbf{b} \Leftrightarrow \mathbf{A}\mathbf{d_x} = \mathbf{b} - (\mathbf{A}\mathbf{x} + \mathbf{y}) = \mathbf{0}$. Thus, $\mathbf{d_x}$ is also a feasible direction at $\mathbf{x}^*$. Since $\mathbf{d_y} = \mathbf{0}$, we have $\mathbf{d_x} \geq \mathbf{0}$ and $\mathbf{c}'\mathbf{d_x} < 0$. In other words, $\mathbf{d_x}$ is a positive cost-reducing feasible direction at $\mathbf{x}^*$. Contradiction. Therefore, the original LP is either infeasible or its optimal cost is $-\infty$.

(d) *Infeasible LP and corresponding big-M problem:*

$$
\begin{array}{llll}
\text{minimize} & x_1 - x_2 + x_3 & \text{minimize} & x_1 - x_2 + x_3 + My_1 + My_2 \\
\text{subject to} & \begin{cases} -x_1 - x_3 = 1 \\ -2x_1 + x_3 = 1 \end{cases} & \text{subject to} & \begin{cases} -x_1 - x_3 + y_1 \qquad = 1 \\ -2x_1 + x_3 \qquad + y_2 = 1 \end{cases} \\
& x_1, x_2, x_3 \geq 0 & & x_1, x_2, x_3, y_1, y_2 \geq 0
\end{array}
$$

The simplex tableau is

$$
\begin{array}{c|c|ccccc}
 & -2M & 1+3M & -1 & 1 & 0 & 0 \\
\hline
y_1 & 1 & -1 & 0 & -1 & 1 & 0 \\
y_2 & 1 & 2 & 0 & 1 & 0 & 1
\end{array}
$$

We can see that the second column indicates a positive cost-reducing feasible direction. However, the original LP is infeasible.

LP with $-\infty$ optimal cost and corresponding big-M problem:

$$\begin{array}{llll} \text{minimize} & -x_1 - x_2 & \text{minimize} & -x_1 - x_2 + My_1 \\ \text{subject to} & x_1 - x_2 = 1 & \text{subject to} & x_1 - x_2 + y_1 = 1 \\ & x_1, x_2 \geq 0 & & x_1, x_2, y_1 \geq 0 \end{array}$$

The simplex tableau is

$$\begin{array}{c|c|ccc} & -M & -1-M & \text{-1+M} & 0 \\ \hline y_1 & 1 & 1 & \text{-1} & 1 \end{array} \Rightarrow \begin{array}{c|c|ccc} & 1 & 0 & \text{-2} & \text{1+M} \\ \hline x_1 & 1 & 1 & \text{-1} & 1 \end{array}$$

We can see that the second column of the last tableau indicates a positive cost-reducing feasible direction, and the original LP has $-\infty$ optimal cost.

Exercise 3.28 $x_i \leq U \quad \forall i = 1, \ldots, n \quad \Rightarrow \sum_{i=1}^{n} x_i \leq U$. Introduce a new variable $x_{n+1} \geq 0$ such that $\sum_{i=1}^{n+1} x_i = U$. Let $\tilde{x}_i = \frac{x_i}{U} \quad \forall i \in \{1, \ldots, n+1\}$. Then $\sum_{i=1}^{n+1} \tilde{x}_i = 1$. Let $\tilde{\mathbf{c}}' = (Uc_1, Uc_2, \ldots, Uc_n, 0)$, $\tilde{\mathbf{A}} = [\mathbf{A} \quad \mathbf{0}]$ and $\tilde{\mathbf{b}} = \frac{1}{U}\mathbf{b}$. Now the LP problem is reformulated as

$$\begin{array}{ll} \text{minimize} & \tilde{\mathbf{c}}'\tilde{\mathbf{x}} \\ \text{subject to} & \tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}} \\ & \displaystyle\sum_{i=1}^{n+1} \tilde{x}_i = 1 \\ & \tilde{\mathbf{x}} \geq 0 \end{array}$$

Since the convexity constraint was derived from the description of the feasible set and we simply rescaled the variables, this problem is equivalent to the original one.

Exercise 3.29 Without loss of generality, let the $m+1$ basic points be $\{(\mathbf{A}_i, c_i) \mid i = 1, \ldots, m+1\}$. Assume otherwise they are affinely dependent. Then the $\mathbb{R}^{m+1}$ vectors $(\mathbf{A}_1 - \mathbf{A}_{m+1}, c_1 - c_{m+1}), \ldots, (\mathbf{A}_m - \mathbf{A}_{m+1}, c_m - c_{m+1})$ are linearly dependent. There are scalers $a_1, \ldots, a_m \in \mathbb{R}$, not all 0, such that

$$\sum_{i=1}^{m} a_i(\mathbf{A}_i - \mathbf{A}_{m+1}, c_i - c_{m+1}) = \mathbf{0}$$

$$\Rightarrow \sum_{i=1}^{m} a_i\mathbf{A}_i - \left(\sum_{i=1}^{m} a_i\right)\mathbf{A}_{m+1} = \mathbf{0}$$

$$\Rightarrow \sum_{i=1}^{m} a_i(\mathbf{A}_i, 1) - \left(\sum_{i=1}^{m} a_i\right)(\mathbf{A}_{m+1}, 1) = \mathbf{0}$$

The last equation implies that the column vectors $(\mathbf{A}_i, 1)'s$ are linearly dependent. Contradiction. Therefore, the $m+1$ basic points are affinely independent.

Exercise 3.30 Without loss of generality, let the $m+1$ basic points be $\{(\mathbf{A}_i, c_i) | i = 1, \ldots, m+1\}$ and $\mathbf{B}$ be the associated basis matrix. Choose a point $(\mathbf{A}_j, c_j)$. In order to calculate the vertical distance from this point to the dual plane, we have to find its vertical projection into the dual plane, say $(\mathbf{A}_j, c_j^*)$. Since $(\mathbf{A}_j, c_j^*)$ is on the dual plane, there are scalars $\lambda_1, \ldots, \lambda_{m+1} \in [0,1]$ such that $\sum_{i=1}^{m+1} \lambda_i (\mathbf{A}_1, c_1) = (\mathbf{A}_j, c_j^*)$ and $\sum_{i=1}^{m+1} \lambda_i = 1$. We can solve the equation $\mathbf{B}\lambda = (\mathbf{A}_j, 1)^T$ to find λ. Since B is invertible, we immediately obtain our solution $\lambda = \mathbf{B}^{-1}(\mathbf{A}_j, 1)^T$. Then

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \ldots & \mathbf{A}_{m+1} \\ c_1 & c_2 & \ldots & c_{m+1} \end{bmatrix} \mathbf{B}^{-1} \begin{bmatrix} \mathbf{A}_j \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_j \\ c_j^* \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{I}_m & \mathbf{0}_{m\times 1} \\ \mathbf{c}_\mathbf{B}^\mathbf{T}\mathbf{B}^{-1} & \end{bmatrix} \begin{bmatrix} \mathbf{A}_j \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_j \\ c_j^* \end{bmatrix}$$

$$\Rightarrow \quad c_j^* = \mathbf{c}_\mathbf{B}^\mathbf{T}\mathbf{B}^{-1}(\mathbf{A}_j, 1)^T$$

Therefore, the vertical distance from the dual plane to $(\mathbf{A}_j, c_j)$ is: $c_j - \mathbf{c}_\mathbf{B}^\mathbf{T}\mathbf{B}^{-1}(\mathbf{A}_j, 1)^T$, which is the reduced cost of x_j as desired.