MATH 170 – PROBLEM SET 3 (DUE TUESDAY FEBRUARY 7)
solutions by Frederick Law

1. (B&T 2.18) Consider a polyhedron $P = \{x : Ax \geq b\}$. Given any $\epsilon > 0$, show that there exists some $\bar{b}$ with the following two properties: (a) The absolute value of every component $b_i - \bar{b}$ is bounded by $\epsilon$. (b) Every basic feasible solution in the polyhedron $P = \{x : Ax \geq b\}$ is nondegenerate.

**Solution:** Intuitively, we shall find such a $b$ by first identifying excessive constraints and perturbing the necessary constraints by some $\epsilon$. That is, if $P = H_1 \cap \cdots \cap H_m$, where $H_i$ are the half-spaces whose intersection gives us $P$, then we identify those $H_j$ which are unnecessary in defining $P$. If $H_j$ is excessive, then $P$ is well defined apart from $H_j$, which means that $\bigcap_{k \neq j} H_k = P$. Let $M$ be the set of all indices of excessive half-spaces. Then $P = \bigcap_{k \in M} H_k$, that is, we can remove all the half-spaces of $M$ and still get the same polyhedron. This is true just because of our construction, since all the half-spaces removed are excessive. If $H_j = \{x \in \mathbb{R}^n : a_j'x \geq b_j\}$, then let $H'_k = \{x \in \mathbb{R}^n : a_k'x \geq b_j - \epsilon\}$. Let $\bar{b}$ be defined by:

$$\bar{b} = \begin{cases} b_j & j \notin M \\ b_j - \epsilon & j \in M \end{cases}$$

Then our new polyhedron is $P' = \bigcap_{k \in M} H_k \cap \bigcap_{j \in M} H'_j$. Note that in $P$, the degenerate basic feasible solutions can be interpreted as having too many hyperplanes, boundary of half-spaces, touching at that point. For example, on a cube in three dimensions, all the points are non-degenerate as they all have 3 hyperplanes touching them, the three facets that connect at a corner. Moreover, every facet of a polyhedron is not excessive, since it serves as a boundary of the polyhedron. Thus, we can interpret the excessive constraints as half-spaces whose hyperplane boundary either touches the polyhedron only at some degenerate solution or at not degenerate solution. Therefore, by perturbing these outward by $\epsilon$, we remove all the degeneracy from our basic feasible solutions. Thus all the basic feasible solutions in $P'$ are nondegenerate. Also, since we have only perturbed the excessive half-spaces by $\epsilon$, it follows that by construction $|b_i - \bar{b}| \leq \epsilon$ for all $i$.

2. (B&T 2.21) Suppose that Fourier-Motzkin elimination is used in the manner described at the end of Section 2.8 to find the optimal cost in a linear programming problem. Show how this approach can be augmented to obtain an optimal solution as well.

**Solution:** To get an optimal solution using the Fourier-Motzkin elimination, we first find the optimal solution. This is done by extending our LP by one variable $x_0$, so we get a new polyhedron in $\mathbb{R}^{n+1}$ defined by $\{(x_0, x) : x \in P, c'x = x_0\}$. Then we use the Fourier Motzkin elimination to project onto the first variable, which gives us $\{x_0 \in \mathbb{R} : \exists x \in P \text{ s.t. } c'x = x_0\}$. Then we minimize over this subset of $\mathbb{R}$ to find our optimal cost, call this $c^*$. To get an optimal solution, we really just project back upwards on $n$ dimensions, by inverting the Fourier Motzkin algorithm. Moreover, to save time in our algorithm, we really only need to project upwards starting at $c^*$. That is, if we imagine our entire process as a mapping $\Phi : P \rightarrow \mathbb{R}$ where $x \in P$ gets sent to $c'x$, then we take the preimage over $c^*$ which is just the level set $\Phi^{-1}(c^*)$ and see what points lie on the level set. This is also the same as taking the plane $\{x \in \mathbb{R}^n : c'x = c^*\}$ and finding where this hyperplane intersects $P$.

3. (B&T 2.22) Let $P$ and $Q$ be polyhedra in $\mathbb{R}^n$. Let $P + Q = \{x + y : x \in P, y \in Q\}$.

(a) Show that $P + Q$ is a polyhedron.

**Solution:** Let us define $M$ as $M = \{(z, x, y) : x \in P, y \in Q, z = x + y\}$. If $P$ is constructed with $n_1$ linear constraints and $Q$ is constructed with $n_2$ linear constraints, then $M$ is constructed with $n_1 + n_2 + n$ linear constraints, where $n$ of them come from
\[ z = x + y \] component wise. Therefore since \( M \) is constructed using linear constraints, \( M \) is a polyhedron in \( \mathbb{R}^n \). Then we use Fourier–Motzkin elimination to reduce to the first \( n \) coordinates: \( \Pi_\mathcal{P}(M) = \{ z \in \mathbb{R}^n : \exists x \in P, y \in Q \text{ s.t. } x + y = z \} \). This can be rewritten as \( \Pi_\mathcal{P}(M) = \{ x + y \in \mathbb{R}^n : x \in P, y \in Q \} = P + Q \). By the Fourier–Motzkin elimination algorithm, we know that \( \Pi_\mathcal{P}(M) \) is a polyhedron, and thus \( P + Q \) is a polyhedron. \( \square \)

(b) Show that every extreme point of \( P + Q \) is the sum of an extreme point of \( P \) and an extreme point of \( Q \).

**Solution:** Suppose not. Then there exists \( x + y \) which is extreme in \( P + Q \) but either \( x \) is not extreme in \( P \) or \( y \) is not extreme in \( Q \) or both. WLOG, suppose that \( x \) is not an extreme in \( P \), \( y \) may or may not be extreme in \( Q \). Since \( x \) is not extreme in \( P \) then that means there exists \( z, z' \in P, \lambda \in [0, 1] \) such that \( x \neq z \) and \( x \neq z' \) and \( x = \lambda x + (1 - \lambda)z' \).

Then we have

\[
x + y = \lambda x + (1 - \lambda)z' + y = \lambda (z + y) + (1 - \lambda)(z' + y)
\]

But now we have written \( x + y \) as a convex combination of \( z + y \) and \( z' + y \), where \( x + y \neq z + y \) and \( z' + y \), since \( x \neq z \) and \( x \neq z' \). Therefore \( x + y \) is not an extreme point. This is a contradiction, so we are done. \( \square \)

4. (B&T 3.2) (Optimality conditions) Consider the problem of minimizing \( c'x \) over a polyhedron \( P \). Prove the following:

(a) A feasible solution \( x \) is optimal if and only if \( c'd \geq 0 \) for every feasible direction \( d \) at \( x \).

**Solution:** First we prove the forward direction. Suppose that \( x \), a feasible solution, is optimal. Then it follows that \( \forall y \in P, c'x \leq c'y \). Suppose \( d \) is an arbitrary feasible direction at \( x \). Then there exists \( \theta > 0 \) such that \( x + \theta d \in P \). Then \( c'x \leq c'(x + \theta d) = c'x + \theta c'd \). Thus \( \theta c'd \geq 0 \), and since \( \theta > 0 \), we divide by \( \theta \) and get \( c'd \geq 0 \).

Now we prove the backward direction. Suppose that \( c'd \geq 0 \) for every feasible direction \( d \) at \( x \). Let \( y \in P \) be arbitrary. Then let \( d = y - x \). Then if we let \( \theta = 1 \), then \( x + \theta d = y \in P \), so \( d \) is a feasible direction at \( x \). So \( c'd \geq 0 \). Thus it follows that \( c'(y - x) \geq 0 \), and so \( c'y = c'x \geq 0 \) and \( c'x \leq c'y \). Since \( y \) was an arbitrary point in \( P \), it follows that \( x \) is an optimal solution. This proves the equality. \( \square \)

(b) A feasible solution \( x \) is the unique optimal solution if and only if \( c'd > 0 \) for every nonzero feasible direction \( d \) at \( x \).

**Solution:** Our argument will be similar to that in part (a). First we prove the forward direction. Suppose that \( x \), a feasible solution, is unique optimal. Then this means that for any \( y \in P \) such that \( y \neq x \), \( c'x < c'y \). Let \( d \) be any nonzero feasible direction at \( x \). Then there exists \( \theta > 0 \) such that \( x + \theta d \in P \). Since \( \theta > 0 \) and \( d \neq 0 \), then it follows that \( x \neq x + \theta d \). Therefore \( c'x < c'(x + \theta d) = c'x + \theta c'd \). Subtracting \( c'x \) from both sides and dividing by \( \theta > 0 \), it follows that \( c'd > 0 \).

Now we prove the backward direction. Suppose that \( c'd > 0 \) for any nonzero feasible direction at \( x \). Let \( y \in P \) be an arbitrary point such that \( y \neq x \). Since \( y \neq x \), then \( d = y - x \neq 0 \). Using \( \theta = 1 \), then \( x + \theta d = y \in P \), so \( d \) is a feasible, nonzero, direction at \( x \). Therefore it follows that \( c'd > 0 \). Thus \( c'(y - x) = c'y - c'x > 0 \). And thus it follows that \( c'y > c'x \). Since \( y \) was any point in \( P \) not equal to \( x \), it follows that \( x \) is the unique optimal solution. This proves the equality. \( \square \)

5. (B&T 3.3) Let \( x \) be an element of the standard form polyhedron \( P = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \} \). Prove that a vector \( d \in \mathbb{R}^n \) is a feasible direction at \( x \) if and only if \( Ad = 0 \) and \( d_i \geq 0 \) for every \( i \) such that \( x_i = 0 \).

**Solution:** We first prove the forward direction. Suppose that \( d \) is feasible at \( x \). Then there exists \( \theta > 0 \) such that \( x + \theta d \in P \). This means \( A(x + \theta d) = Ax + \theta Ad = b \). Since \( x \in P \), then
\[ \mathbf{Ax} = \mathbf{b}, \text{ thus subtracting } \mathbf{b} \text{ from both sides, and dividing by } \theta > 0, \text{ it follows that } \mathbf{Ad} = \mathbf{0}. \text{ If } i \text{ is an index such that } x_i = 0, \text{ and since } x + \theta \mathbf{d} \geq 0, \text{ then } x_i + \theta d_i = \theta d_i \geq 0. \text{ Dividing by } \theta > 0, \text{ it follows that } d_i \geq 0 \text{ for any index } i \text{ such that } x_i \geq 0. \]

Now we prove the backward direction. Suppose \( \mathbf{d} \in \mathbb{R}^n \) such that \( \mathbf{Ad} = \mathbf{0} \) and \( d_i \geq 0 \) for any index \( i \) such that \( x_i = 0 \). Let us choose \( \theta^* \) satisfying

\[
0 < \theta^* < \inf \left\{ \frac{x_j}{d_j} : x_j > 0, d_j < 0 \right\}
\]

Since \( x \in P \), then \( x_i \geq 0 \) for all \( i \). If \( x_i = 0 \), then \( x_i + \theta d_i = \theta d_i \geq 0 \), since \( d_i \geq 0 \) and \( \theta > 0 \). If \( x_i > 0 \), there are two cases. If \( d_i \geq 0 \), then \( x_i + \theta d_i \geq 0 \). If \( d_i < 0 \), then \( \theta^* < -\frac{x_i}{d_i} \), and multiplying through by \( d_i < 0 \), we get \( \theta^* d_i > -x_i \), and thus \( x_i + \theta^* d_i \geq 0 \). Thus \( x + \theta^* \mathbf{d} \geq 0 \).

Also, \( \mathbf{A}(x + \theta^* \mathbf{d}) = \mathbf{Ax} + \theta^* \mathbf{Ad} = \mathbf{Ax} = \mathbf{b} \) since \( x \in P \). Thus it follows that \( x + \theta^* \mathbf{d} \in P \). This proves the equality, and we are done.

6. (B&T 3.5) Let \( P = \{ x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1, x \geq 0 \} \) and consider the vector \( x = (0, 0, 1) \). Find the set of feasible directions at \( x \).

**Solution:** We find the set of feasible directions by working with the definition of feasible directions. We say that \( \mathbf{d} \) is feasible at \( x \) if there exists some \( \theta > 0 \) such that \( x + \theta \mathbf{d} \in P \). Let \( \mathbf{d} = (d_1, d_2, d_3) \). Then \( x + \theta \mathbf{d} = (\theta d_1, \theta d_2, 1 + \theta d_3) \). To require this to be in \( P \), we need that \( \theta d_1 + \theta d_2 + d_3 + 1 = 0 \), which means \( d_1 + d_2 + d_3 = 0 \), and thus \( d_3 = -d_1 - d_2 \). We also require \( d_1, d_2 \) to be non-negative, since then for any \( \theta > 0 \), \( \theta d_1, d_2 \geq 0 \). Lastly, we shall require \( 1 + \theta d_3 \geq 0 \). To do this, we shall choose \( \theta \) so that \( 1 + \theta d_3 = 0 \). Then \( \theta = \frac{1}{1 + \theta d_3} \) and \( \theta = \frac{1}{d_1 + d_2} \). Note this \( \theta \) will always be positive, unless \( d_1 = d_2 = 0 \). But in that case, \( d_3 = 0 \), and \( \mathbf{d} = \mathbf{0} \), which is always a feasible direction. Thus our set of feasible directions is

\[
F = \{ (d_1, d_2, -d_1 - d_2) : d_1 \geq 0, d_2 \geq 0 \}
\]

As a safe mental check, if \( d_1 = d_2 = 0 \), then \( \mathbf{d} = \mathbf{0} \), and for \( \theta = 1 \), \( x + \theta \mathbf{d} = x \in P \). If either \( d_1 \) or \( d_2 > 0 \), then let \( \theta = \frac{1}{d_1 + d_2} > 0 \), and then \( x + \theta \mathbf{d} = \left( \frac{d_1}{d_1 + d_2}, \frac{d_2}{d_1 + d_2}, 0 \right) \in P \), since \( \frac{d_1}{d_1 + d_2} + \frac{d_2}{d_1 + d_2} = 1, \text{ and } \frac{d_1}{d_1 + d_2} \geq 0 \) for \( i = 1, 2 \).

7. (B&T 3.6) (Conditions for a unique optimum) Let \( x \) be a basic feasible solution associated with some basis matrix \( \mathbf{B} \). Prove the following:

(a) If the reduced cost of every nonbasic variable is positive, then \( x \) is the unique optimal solution.

**Solution:** Suppose that \( \bar{c}_j > 0 \) for every nonbasic variable \( j \). Let \( y \in P \) be arbitrary such that \( y \neq x \). Then \( \mathbf{Ax} = \mathbf{Ay} = \mathbf{b} \). Thus it follows that \( \mathbf{A}(y - x) = \mathbf{0} \). Let \( \mathbf{d} = y - x \). Then using \( \theta = 1, x + \theta \mathbf{d} = y \in P \), so \( \mathbf{d} \) is a feasible direction at \( x \) and \( \mathbf{Ad} = \mathbf{0} \). We can rewrite this as \( \mathbf{Bd}_B + \sum_{j \in I} A_j d_j = 0 \) where \( I \) is the set of nonbasic indices. Since \( \mathbf{B} \) is invertible, then \( d_B = -\sum_{j \in I} \mathbf{B}^{-1} A_j d_j \).

\[
c' \mathbf{d} = c_B' \mathbf{d}_B + \sum_{j \in I} c_j d_j = \sum_{j \in I} (c_j - c_B' \mathbf{B}^{-1} A_j) d_j = \sum_{j \in I} \bar{c}_j d_j
\]

Since \( y \neq x \), there must exist some \( j \in I \) such that \( y_j \neq x_j \), where \( x_j = 0 \) for all \( j \in I \) since \( x \) is a basic feasible solution. If \( y \in P \) such that \( y_j = 0 \) for all \( j \in I \), then \( \mathbf{Ay} = \mathbf{By} = \mathbf{b} \). Then \( \mathbf{y} = -\mathbf{B}^{-1} \mathbf{b} = x_B \), and so \( y = x \). Therefore there must exist some nonbasic index on which \( x \) and \( y \) disagree. Let \( k \in I \) be such an index. Since \( y \in P \), then \( y_k \geq 0 \), and since \( y_k \neq x_k \), then \( y_k > 0 \). Since \( d_j = y_j - x_j \) and since \( x_j = 0 \) for all \( j \in I \):

\[
c' \mathbf{d} = \sum_{j \in I} c_j d_j = \sum_{j \in I} \bar{c}_j y_j \geq \bar{c}_k y_k > 0
\]
where we use the fact that \( \bar{e}_j \geq 0 \) for all \( j \in I \) and \( y_j \geq 0 \) for all \( j \in I \) since \( y \in P \). Thus we have \( c'(y - x) > 0 \) and thus \( c'x < c'y \). Since this holds for all \( y \neq x \), it follows that \( x \) is the unique optimal solution.

(b) If \( x \) is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.

**Solution**: Suppose not. Then there exists an index \( m \in I \), borrowing notation from part (a), such that \( \bar{e}_m \leq 0 \). Let \( d \) be the \( m \)th basic direction. We know from our results in class and in Bertsimas and Tsitsiklis that \( d \) is always a feasible direction at \( x \) since \( x \) is nondegenerate. Then there exists \( \theta > 0 \) such that \( x + \theta d \in P \). Then

\[
c'd = \sum_{j \in I} \bar{e}_j d_j = \bar{e}_k \leq 0
\]

since \( d \) is the \( k \)th basic direction, so \( d_k = 1 \) and \( d_j = 0 \) for \( j \in I, j \neq k \). Therefore \( c'd \leq 0 \). Choosing \( \theta > 0 \) sufficiently small so that \( x + \theta d \in P \), we get \( \theta c'd \leq 0 \) which implies that \( c'(x + \theta d) = c'x + \theta c'd \leq c'x \). But \( x + \theta d \in P \), which means that \( x \) is not the unique optimal solution. This is a contradiction, which completes the proof.

8. (B&T 3.10) Show that if \( n - m = 2 \), then the simplex method will not cycle, no matter which pivoting rule is used.

**Solution**: Note that there is a lot of possible choices in the simplex method, thus if this fact is indeed true it, then the face that \( n - m = 2 \) must do something eliminate our possible choices in directions to move when executing the simplex method. One possible explanation could be that when \( n - m = 2 \), then either the polyhedron is unbounded, in which case simplex terminates at optimal cost being \(-\infty\) or the polyhedron is bounded and it happens that in the polytope all the basic feasible solutions are nondegenerate, and then the simplex method on this polytope will have to terminate.