Math 170: Homework 1

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Exercises 1.1, 1.4, 1.7, 1.8, 1.12, 1.14, 1.19

Problem 1.1 Suppose that a function $f : \mathbb{R}^n \to \mathbb{R}$ is both concave and convex. Prove that f is an affine function.
Solution. Let $f(\mathbf{0}) = a$ and let the function $g = f - a$. Then $g(\mathbf{0}) = 0$ and g is also both concave and convex. Let $\lambda \in [0, 1]$, and let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Since g is convex, we have that that $g(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \lambda g(\boldsymbol{x}) + (1-\lambda)g(\boldsymbol{y})$. Since g is concave, we also have that $g(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \lambda g(\boldsymbol{x}) + (1-\lambda)g(\boldsymbol{y})$. Hence, we have $g(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) = \lambda g(\boldsymbol{x}) + (1-\lambda)g(\boldsymbol{y})$. This implies that $g(\lambda\boldsymbol{x}) = \lambda g(\boldsymbol{x})$ for $\lambda \in [0, 1]$ and all $\boldsymbol{x} \in \mathbb{R}^n$, which implies that g is linear. Hence, f is affine.

Problem 1.4 Consider the problem

$$\text{minimize } 2x_1 + 3\,|x_2 - 10|$$

$$\text{subject to } \ |x_1 + 2| + |x_2| \leq 5,$$

and reformulate it as a linear programming problem.
Solution. We are looking to minimize the cost function $2x_1 + 3 \cdot \max\{x_2 - 10, -x_2 + 10\}$. We replace the max function with the variable z_1 and add in the restrictions

$$z_1 \geq x_2 - 10, z_1 \geq -x_2 + 10$$

so that the minimum possible value of z_1 is $|x_2 - 10|$. Similarly, we add in z_2 and z_3, adding the restrictions

$$z_2 \geq x_1 + 2, z_2 \geq -x_1 - 2, z_3 \geq x_2, z_3 \geq -z_3$$

so that the original restriction becomes $z_2 + z_3 \leq 5$. Our linear programming problem is now

$$\text{minimize } 2x_1 + 3z_1$$

$$\text{subject to } \ z_1 \geq x_2 - 10, z_1 \geq -x_2 + 10, z_2 \geq x_1 + 2, z_2 \geq -x_1 - 2, z_3 \geq x_2, z_3 \geq -x_2, z_2 + z_3 \leq 5.$$

Problem 1.7 Suppose that Z is a random variable taking values in the set $0, 1, \ldots, K,$, with probabilities $p_0, p_1, \ldots, p_K$, respectively. We are given the

values of the first two moments $E[Z] = \sum_{k=0}^{K} kp_k$ and $E[Z^2] = \sum_{k=0}^{K} k^2 p_k$ of Z and we would like to obtain upper and lower bounds on the value of the fourth moment $E[Z^4] = \sum_{k=0}^{K} k^4 p_k$ of Z. Show how linear programming can be used to approach this problem.
Solution. We let $p_0, \ldots, p_K$ be the variables in our linear program. We would then have two linear programs: one to minimize $E[Z^4] = \sum_{k=0}^{K} k^4 p_k$ and the other to maximize $E[Z^4] = \sum_{k=0}^{K} k^4 p_k$. These would give us our upper and lower bounds. We use the known moments to give us two restrictions on our variables. We also know that the p_i sum to 1, and that each $p_i \geq 0$. This gives us $K+4$ restrictions to use in our two linear programs.

Problem 1.8 Consider a road divided into n segments that is illuminated by m lamps. Let p_j be the power of the jth lamp. The illumination I_i of the ith segment is assumed to be $\sum_{j=1}^{m} a_{ij} p_j$, where a_{ij} are known coefficients. Let $I_I^\star$ be the desired illumination of road i.
We are interested in choosing the lamp powers p_j so that the illuminations are close to the desired illuminations $I_i^\star$. Provide a reasonable linear programming formulation of this problem.
Solution. We are interested in minimizing the differences between the illumination I_i of each segment and the desired illumination of that segment, $I_i^\star$. So, our linear program will to be to

minimize $\sum_{i=1}^{n} |I_i - I_i^\star|$, where I_i is given by the summation in the problem statement, and our variables are then the p_j,

subject to the condition that each $p_j \geq 0$.

Problem 1.12 Consider a set P described by linear inequality constraints, that is, $P = \{x \in \mathbb{R}^n | \boldsymbol{a}_i' \boldsymbol{x} \leq b_i, i = 1, \ldots, m\}$. A ball with center $\boldsymbol{y}$ and radius r is defined as the set of all points within the distance r from $\boldsymbol{y}$. We are interested in finding a ball with the largest possible radius, which is entirely contained within the rest P. Provide a linear programming formulation of this problem.
Solution. We are trying to pick a point $\boldsymbol{y}$ that maximizes the shortest distance from $\boldsymbol{y}$ to the boundary of the set P. The dot product of our potential point $\boldsymbol{y}$ with the normal vector to each hyperplane defining the boundary will help us calculate this distance. Our linear program is thus

$$\text{minimize } \max_i \{-|\boldsymbol{a}_i' \boldsymbol{y} - b_i|\}$$

$$\text{subject to } \boldsymbol{a}_i' \boldsymbol{y} \leq b_i \text{ for all } i.$$

Problem 1.14
Solution.

(a) Let a be the number of units of the first product and let b be the number of units of the second product. The desired linear program is then

$$\text{maximize } 3a + 2.4b$$

$$\text{subject to } 3a + 4b \leq 20,000,$$
$$3a - (6 \cdot 0.45)a + 2b - (5.4 \cdot 0.3)b \leq 4000, a \geq 0, b \geq 0.$$

(b) **Solution**. Drawing was done by hand. Net income is only restricted by machine hours in this case, and the net income is maximized by only producing product a, generating $20,000.

(c) **Solution**. Yes, this will help the company.

Problem 1.19 Suppose that we are given a set of vectors in $\mathbb{R}^n$ that form a basis, and let $\boldsymbol{y}$ be an arbitrary vector in $\mathbb{R}^n$. We wish to express $\boldsymbol{y}$ as a linear combination of the basis vectors. How can this be accomplished?

Solution. We let $a_1, \ldots, a_n$ denote the coefficients we are trying to find, and let $\boldsymbol{b}^1, \ldots, \boldsymbol{b}^n$ denote the given basis vectors. We would like to impose the constraints $\sum_{i=1}^n a_i b_j^i = y_j$ for all j, where b_j^i is the jth component of the ith basis vector. These equations will have a unique solution. Thus, we can simply minimize a constant function (such as 0) subject to these constraints in order to find the desired coefficients.