Problem 1.1 Suppose that a function $f : \mathbb{R}^n \to \mathbb{R}$ is both concave and convex. Prove that $f$ is an affine function.

Solution. Let $f(0) = a$ and let the function $g = f - a$. Then $g(0) = 0$ and $g$ is also both concave and convex. Let $\lambda \in [0, 1]$, and let $x, y \in \mathbb{R}^n$. Since $g$ is convex, we have that $g(\lambda x + (1 - \lambda) y) \leq \lambda g(x) + (1 - \lambda) g(y)$. Since $g$ is concave, we also have that $g(\lambda x + (1 - \lambda) y) \geq \lambda g(x) + (1 - \lambda) g(y)$. Hence, we have $g(\lambda x + (1 - \lambda) y) = \lambda g(x) + (1 - \lambda) g(y)$. This implies that $g(\lambda x) = \lambda g(x)$ for $\lambda \in [0, 1]$ and all $x \in \mathbb{R}^n$, which implies that $g$ is linear. Hence, $f$ is affine.

Problem 1.4 Consider the problem

$$\text{minimize $2x_1 + 3|x_2 - 10|$}$$

subject to $|x_1 + 2| + |x_2| \leq 5$,

and reformulate it as a linear programming problem.

Solution. We are looking to minimize the cost function $2x_1 + 3 \cdot \max\{x_2 - 10, -x_2 + 10\}$. We replace the max function with the variable $z_1$ and add in the restrictions

$$z_1 \geq x_2 - 10, z_1 \geq -x_2 + 10$$

so that the minimum possible value of $z_1$ is $|x_2 - 10|$. Similarly, we add in $z_2$ and $z_3$, adding the restrictions

$$z_2 \geq x_1 + 2, z_2 \geq -x_1 - 2, z_3 \geq x_2, z_3 \geq -x_2$$

so that the original restriction becomes $z_2 + z_3 \leq 5$. Our linear programming problem is now

$$\text{minimize $2x_1 + 3z_1$}$$

subject to $z_1 \geq x_2 - 10, z_1 \geq -x_2 + 10, z_2 \geq x_1 + 2, z_2 \geq -x_1 - 2, z_3 \geq -x_2, z_3 \geq x_2, z_3 \geq -2, z_3 \geq -z_3 \leq 5$.

Problem 1.7 Suppose that $Z$ is a random variable taking values in the set $0, 1, \ldots, K$, with probabilities $p_0, p_1, \ldots, p_K$, respectively. We are given the
values of the first two moments \( E[Z] = \sum_{k=0}^{K} kp_k \) and \( E[Z^2] = \sum_{k=0}^{K} k^2 p_k \) of \( Z \) and we would like to obtain upper and lower bounds on the value of the fourth moment \( E[Z^4] = \sum_{k=0}^{K} k^4 p_k \) of \( Z \). Show how linear programming can be used to approach this problem.

**Solution.** We let \( p_0, \ldots, p_K \) be the variables in our linear program. We would then have two linear programs: one to minimize \( E[Z^4] = \sum_{k=0}^{K} k^4 p_k \) and the other to maximize \( E[Z^4] = \sum_{k=0}^{K} k^4 p_k \). These would give us our upper and lower bounds. We use the known moments to give us two restrictions on our variables. We also know that the \( p_i \) sum to 1, and that each \( p_i \geq 0 \). This gives us \( K + 4 \) restrictions to use in our two linear programs.

**Problem 1.8** Consider a road divided into \( n \) segments that is illuminated by \( m \) lamps. Let \( p_j \) be the power of the \( j \)-th lamp. The illumination \( I_i \) of the \( i \)-th segment is assumed to be \( \sum_{j=1}^{m} a_{ij} p_j \), where \( a_{ij} \) are known coefficients. Let \( I^*_i \) be the desired illumination of road \( i \).

We are interested in choosing the lamp powers \( p_j \) so that the illuminations are close to the desired illuminations \( I^*_i \). Provide a reasonable linear programming formulation of this problem.

**Solution.** We are interested in minimizing the differences between the illumination \( I_i \) of each segment and the desired illumination of that segment, \( I^*_i \). So, our linear program will be to

\[
\text{minimize } \sum_{i=1}^{n} |I_i - I^*_i|, \quad \text{where } I_i \text{ is given by the summation in the problem statement, and our variables are then the } p_j,
\]

subject to the condition that each \( p_j \geq 0 \).

**Problem 1.12** Consider a set \( P \) described by linear inequality constraints, that is, \( P = \{ x \in \mathbb{R}^n | a'_i x \leq b_i, i = 1, \ldots, m \} \). A ball with center \( y \) and radius \( r \) is defined as the set of all points within the distance \( r \) from \( y \). We are interested in finding a ball with the largest possible radius, which is entirely contained within the rest \( P \). Provide a linear programming formulation of this problem.

**Solution.** We are trying to pick a point \( y \) that maximizes the shortest distance from \( y \) to the boundary of the set \( P \). The dot product of our potential point \( y \) with the normal vector to each hyperplane defining the boundary will help us calculate this distance. Our linear program is thus

\[
\text{minimize } \max_i \{|a'_i y - b_i|\}
\]

subject to \( a'_i y \leq b_i \) for all \( i \).

**Problem 1.14**

**Solution.**

(a) Let \( a \) be the number of units of the first product and let \( b \) be the number of units of the second product. The desired linear program is then

\[
\text{maximize } 3a + 2.4b
\]
subject to $3a + 4b \leq 20,000$,  
$3a - (6 \cdot 0.45)a + 2b - (5.4 \cdot 0.3)b \leq 4000$, $a \geq 0$, $b \geq 0$.

(b) **Solution.** Drawing was done by hand. Net income is only restricted by machine hours in this case, and the net income is maximized by only producing product $a$, generating $20,000$.

(c) **Solution.** Yes, this will help the company.

**Problem 1.19** Suppose that we are given a set of vectors in $\mathbb{R}^n$ that form a basis, and let $y$ be an arbitrary vector in $\mathbb{R}^n$. We wish to express $y$ as a linear combination of the basis vectors. How can this be accomplished?

**Solution.** We let $a_1, \ldots, a_n$ denote the coefficients we are trying to find, and let $b_1, \ldots, b_n$ denote the given basis vectors. We would like to impose the constraints \( \sum_{i=1}^{n} a_i b_j = y_j \) for all $j$, where $b_j^i$ is the $j$th component of the $i$th basis vector. These equations will have a unique solution. Thus, we can simply minimize a constant function (such as 0) subject to these constraints in order to find the desired coefficients.