Score 7

MATH 170 HW#2

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Exercise 2.1 (a) Consider the polar coordinate system. Let $x = r\cos t$, $y = r\sin t$ $r \geq 0$, $t \in [0, 2\pi]$. Then $x\cos\theta + y\sin\theta \leq 1 \Leftrightarrow r\cos(\theta - t) \leq 1$ and $x \geq 0, y \geq 0 \Leftrightarrow t \in [0, \frac{\pi}{2}]$. Since the inequality must hold for all $\theta \in [0, \frac{\pi}{2}]$, we have $r \leq 1$. Therefore, the set actually is a quarter of a unit circle and hence is not a polyhedron. 1

(b)

$$x^2 - 8x + 15 \leq 0 \quad \Leftrightarrow \quad \begin{cases} x \leq 5 \\ x \geq 3 \end{cases}$$

Thus, the set is a polyhedron of the form $\{x \in \mathbb{R} | x \geq 3,\ x \leq 5\}$.

(c) Empty set is a polyhedron. An example is $\{x \in \mathbb{R} | x \geq 1,\ x \leq 0\}$.

Exercise 2.3 Let A be a $m \times n$ matrix, $x, u \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Then the polyhedron is $\{x \in \mathbb{R}^n |\ Ax = b,\ 0 \leq x \leq u\}$. The procedure for finding a basic solution is as follows.

(1) Choose m linearly independent columns of A, $A_{B(1)}, \dots, A_{B(m)}$.

(2) Let $x_i = 0$ or u_i for all $i \neq B(1), \dots, B(m)$.

(3) Solve $Ax = b$ for the unknown variables $x_{B(1)}, \dots, x_{B(m)}$. Now we prove a theorem justifying this procedure.

Theorem. *Consider the constraints $Ax = b$, $x \geq 0$ and $x \leq u$. Assume the $m \times n$ matrix A has linearly independent rows. A vector $x \in \mathbb{R}^n$ is a basic solution if and only if we have $Ax = b$, and there exist indices $B(1), \dots, B(m)$ such that:*

(1) The columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent;

(2) If $i \neq B(1), \dots, B(m)$, then $x_i = 0$ or u_i.

Proof. Suppose $x \in \mathbb{R}^n$ satisfies both conditions. Then we have

$$Ax = \sum_{i=1}^{m} A_{B(i)}x_{B(i)} + \sum_{i \neq B(1),\dots,B(m)} A_i x_i = b$$

$$\Leftrightarrow \sum_{i=1}^{m} A_{B(i)}x_{B(i)} = b - \sum_{i \neq B(1),\dots,B(m)} A_i x_i$$

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Since $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent, this equation uniquely determine x_i for $i = B_1, \dots, B_m$. Thus, the system of equations formed by the active constraints has a unique solution. By Theorem 2.2, there are n linearly independent active constraints. Therefore, x is a basic solution. For the converse, consider x being a basic solution. We will show there exist indices $B(1), \dots, B(m)$ satisfying both conditions. Let $B(1), \dots, B(k)$ be all indices that $x_{B(i)} \neq 0$ and $x_{B(i)} \neq u_{B(i)}$ for all $i = 1, \dots, k$. Since x is a basic solution, the system of equations formed by the active constraints has a unique solution. (i.e. $\sum_{i=1}^{k} A_{B(i)}x_{B(k)} = b - \sum_{i \neq B(1),\dots,B(k)} A_i x_i$ has a unique solution.) This implies that $A_{B(1)}, \dots, A_{B(k)}$ are linearly independent and $k \leq m$. Since rank(A)=m, we can choose $m-k$ more columns from A to extend the family such that $A_{B(1)},\ldots,A_{B(m)}$ are linearly independent. Moreover, if $i\neq B(1),\ldots,B(m)$, then clearly $i\neq B(1),\ldots,B(k)$ and hence $x_i=0$ or u_i.

$\square$

Exercise 2.4 A polyhedron in standard form has at least one extreme point because it does not contain a line. However, this is not true for general polyhedra. For example, the halfspace $\{(x,y)\in\mathbb{R}^2|\,x+y\geq 1\}$ is a trivial polyhedron with no extreme point.

Exercise 2.6 (a) If $n\leq m$, then the result is trivial. Now suppose $n>m$. Let $y\in C$. Consider the polyhedron $\Lambda=\{(\lambda_1,\ldots,\lambda_n)\in\mathbb{R}^n|\,\sum_{i=1}^n\lambda_iA_i=y,\quad \lambda_1,\ldots,\lambda_n\geq 0\}$. This is a polyhedron in standard form, so by theorem there exists a basic feasible solution $\lambda^*=(\lambda_1^*,\ldots,\lambda_n^*)$. Note that we can have at most m linearly independent vectors out of the family A_i. Thus, a basic feasible solution has at least $n-m$ zero components, which means there are at most m non-zero components in λ^*. Thus, $\sum_{i=1}^n\lambda_i^*A_i$ is an expression of y with at most m of the coefficients being non-zero.
(b) If $n\leq m+1$, then the result is trivial. Now suppose $n>m+1$. Let $y\in P$. Consider the polyhedron $\Lambda=\{(\lambda_1,\ldots,\lambda_n)\in\mathbb{R}^n|\,\sum_{i=1}^n\lambda_iA_i=y,\,\sum_{i=1}^n\lambda_i=1,\quad\lambda_1,\ldots,\lambda_n\geq 0\}$.This is a polyhedron in standard form, so by theorem there exists a basic feasible solution $\lambda^*=(\lambda_1^*,\ldots,\lambda_n^*)$. Note that there are $m+1$ equality constraints in the polyhedron, so we can have at most $m+1$ linearly independent equality constraints. Thus, the basic feasible solution have at least $n-(m+1)$ zero components, which means λ^* has at most $m+1$ non-zero components. Thus, $\sum_{i=1}^n\lambda_i^*A_i$ is an expression of y with at most $m+1$ of the coefficients being non-zero.

Exercise 2.7 Assume otherwise the vector $g_1,\ldots,g_k$ do not span $\mathbb{R}^n$. Then, they are all in a proper subspace of $\mathbb{R}^n$. Hence, there exists a non-zero vector $d\in\mathbb{R}^n$ such that $d'g_i=0\quad\forall i=1,\ldots,k$. Let x_0 be an element of the polyhedron represented by $\{x\in\mathbb{R}^n|\,g_i'x\geq h_i,\,i=1,\ldots,k\}$ and $\{x\in\mathbb{R}^n|\,a_i'x\geq b_i,\,i=1,\ldots,m\}$. Consider the line $x_0+\lambda d,\ \lambda\in\mathbb{R}$. Note that $g_i'(x_0+\lambda d)=g_i'x_0\geq h_i\quad\forall i=1,\ldots,k$. Thus, the line is contained in the polyhedron. Hence, we have $a_i'(x+\lambda d)\geq b_i,\,\forall\lambda\in\mathbb{R}\,\forall i=1,\ldots,m$. Clearly, d must be orthogonal to all the vectors a_i's. However, the vectors $a_1,\ldots,a_m$ span $\mathbb{R}^n$, and hence $d=0$. Contradiction. We conclude that the vector $g_1,\ldots,g_k$ also span $\mathbb{R}^n$.

Exercise 2.9 (a) Assume otherwise the basic solution is not degenerate. Then the basic solution have $n-m$ zero components. This uniquely determine m non-zero components, which correspond to a unique choice of basis. Contradiction. Therefore, a basic solution with two different bases must be degenerate.
(b) No, a degenerate basic solution does not necessarily correspond to two different bases. Consider the polyhedron in standard form $\{(x,y)\in\mathbb{R}^2|\,x+y\geq 0,\,x-y\geq 0\quad x,y\geq 0\}$. This is a polyhedron that contains only one point $(0,0)$ which is also a degenerate basic solution (all constraints are active). However, clearly there is only one choice of basis.
(c) No, it is not true. Consider the example in (b). The polyhedron contains only one basic solution, and hence no adjacent basic solution exists.

Exercise 2.10 (a) No, it is not true. Consider the standard form polyhedron $\{(x,y,z)\in\mathbb{R}^3|\,x+y=2,\,y+z=1\quad x,y,z\geq 0\}$. There are 3 basic solutions $(2,0,1),(0,2,-1),(1,1,0)$.
(b) No, it is not true. Consider the LP problem that minimize $-x+y$ over the standard form polyhedron $\{(x,y)\in\mathbb{R}^2|\,-x+y=0,\quad x,y\geq 0\}$. Then every point in the

polyhedron is an optimal solution, and clearly the polyhedron is not bounded.
(c) No, it is not true. Consider again the LP problem in (b). Then $(1,1)$ is an optimal solution with more than 1 positive variables.
(d) Yes, it is true. Let x, y be two distinct optimal solutions and d be the corresponding optimal value. Consider the convex combination of x, y: $\lambda x + (1-\lambda)y$, $\lambda \in [0,1]$. For all $\lambda \in [0,1]$, $c'[\lambda x + (1-\lambda)y] = \lambda d + (1-\lambda)d = d$. Also, since the polyhedron is a convex set, every convex combination of x, y is in the polyhedron. Thus, we have uncountably many optimal solutions $\lambda x + (1-\lambda)y$, $\lambda \in [0,1]$.
(e) No, it is not true. Consider again the counterexample in (b). The polyhedron has only one basic feasible solution but infinitely many optimal solutions.
(f) Yes, it is true. Introduce new variables $z, s_1, s_2 \in \mathbb{R}$ with constraints $c'x + s_1 = z$, $d'x + s_2 = z$, and $z, s_1, s_2 \geq 0$. Then we transform the LP problem into minimizing z over a standard form polyhedron $P' = \{(x_1, \ldots, x_n, z, s_1, s_2) \in \mathbb{R}^{n+3} | Ax = b,\ c'x + s_1 - z = 0,\ d'x + s_2 - z = 0,\ x, z, s_1, s_2 \geq 0\}$ where $x = (x_1, \ldots, x_n)$. Note that there are $n+3$ variables and $m+2$ equality constraints. Let A' be the $m+2$ by $n+3$ matrix of all equality constraints of P'. Since the matrix A has linearly independent rows and the new constraints $c'x + s_1 - z = 0$, $d'x + s_2 - z = 0$ involve new variables respectively, the matrix A' also has linearly independent rows. The LP problem has an optimal solution, so there is an extreme point $x' = (x_1', \ldots, x_n', z', s_1', s_2')$ of P' that is an optimal solution. We now claim $x^* = (x_1', \ldots, x_n') \in \mathbb{R}^n$ is an optimal solution which is an extreme point of P. Obviously, $x^* \in P$. Note that $A'_{(n+1)} + A'_{(n+2)} = A'_{(n+3)}$, so we can choose at most 2 linearly independent columns from $A'_{(n+1)}, A'_{(n+2)}, A'_{(n+3)}$ when constructing a basic solution. That said, we have to choose at least m columns from the first n columns of A. However, we can choose at most m linearly independent columns from the first n columns because rank(A)=m. Therefore, every extreme point of P' has a corresponding basis consisting m vectors from the first n columns of A'. This means x^* has a corresponding basis consisting exactly m vectors from the columns of A. Thus, x^* is an extreme point of P and clearly it is also an optimal solution to the LP problem.