Please start by writing your name and your student ID on the cover of your blue book. This exam is closed book. Do not use any notes, calculators, cell phones etc. You must show all your work to get credit. There are ten problems, each worth 10 points, for a total of 100 points.

(1) Let \( u \) be the class of 11 in \( \mathbb{Z}/23\mathbb{Z} \). Determine \( u^{-1} \) and \( u + u^{-1} \).

(2) What is the smallest order of a non-abelian group?

(3) How many units does the ring \( \mathbb{Z}/60\mathbb{Z} \) have?

(4) Determine the 11th cyclotomic polynomial \( \Phi_{11}(x) \).

(5) For which values of \( a \) in \( \mathbb{F}_5 \) is the ring \( \mathbb{F}_5[x]/\langle x^3 + 2x^2 + a \rangle \) a field?

(6) Consider the ideal \( I = \langle x^2 + y^2 - 1, xy + 2 \rangle \) in \( \mathbb{Q}[x, y] \). Find a polynomial \( f(x) \) which generates the principal ideal \( I \cap \mathbb{Q}[x] \) in \( \mathbb{Q}[x] \).

(7) Prove or disprove: If \( H_1 \) and \( H_2 \) are subgroups of a finite group \( G \) then their product \( H_1 \cdot H_2 \) is a subgroup of \( G \).

(8) Prove or disprove: If \( I \) is an ideal in a principal ideal domain \( R \) then every ideal in the quotient ring \( R/I \) is principal.

(9) Prove or disprove: There exists a term ordering such that the subset \( \{ x^3 + y^2, x^2 + y \} \) of \( \mathbb{Q}[x, y] \) is a Gröbner basis.

(10) Prove or disprove: If \( R \) is the quotient ring \( \mathbb{Q}[x, y, z]/\langle x \cdot z - y^2 \rangle \) then every irreducible element in \( R \) is prime.