Antrag auf Förderung eines Einstein Visiting Fellows

Bernd Sturmfels

Anlage B: Darstellung der Bedeutung des Einstein Visiting Fellows für die gastgebende Einheit

1. Zielsetzung, Relevanz, Fragestellung

The Berlin-based group of Bernd Sturmfels will pursue four main research directions in applied and computational algebraic geometry: *Maximum Likelihood Geometry*, *Euclidean Distance Optimization*, *Convex Algebraic Geometry*, and *Classical Moduli Spaces*. These topics are described in detail below. Twenty concrete research problems will be stated. These problems serve as focus for the group. In addition, Sturmfels plans to work on a new book project. The tentative title, *Nonlinear Algebra*, is the topic of a new graduate course he will develop and teach as a short course while in Berlin.

In the text below, the articles that are cited with numbers, like "[141]" or "[212]", can be found in Sturmfels' list of journal publications. That list is attached. His paper can also be found on-line at

http://math.berkeley.edu/~bernd/articles.html

All referenced items that are cited with letters, like "[DR]" or "[DSS]", are listed in Section 3 below.

Maximum Likelihood Geometry

Consider a discrete random variable on a finite set of states, here taken to be $\{0, 1, \ldots, n\}$. A statistical model for that random variable on $n+1$ states is a subset $\mathcal{M}$ of the probability simplex

$$\Delta_n \quad = \quad \big\{(p_0, p_1, \ldots, p_n) \in \mathbf{R}^{n+1} \,:\, p_0+p_1+\cdots+p_n = 1 \text{ and } p_i \geq 0 \text{ for } i = 0, 1, \ldots, n\big\}.$$

If we collect N i.i.d. samples then these are summarized in a non-negative integer vector $u = (u_0, u_1, \ldots, u_n)$, where u_i is the number of samples with $X = i$. Thus $u_0 + u_1 + \cdots + u_n = N$. Maximum likelihood estimation (MLE) is the problem of finding the probability distribution p in the model $\mathcal{M}$ that best explains the given data u. This amounts to solving the optimization problem

$$\text{Maximize } \sum_{i=0}^{n} u_i \cdot \log(p_i) \quad \text{subject to } p \in \mathcal{M}. \tag{1}$$

In algebraic statistics [DSS] one often considers models of the form $\mathcal{M} = X \cap \Delta_n$, where $X \subset \mathbf{P}^n$ is a projective variety, defined by homogeneous polynomial equations in $p_0, \ldots, p_n$. The *ML degree* of X, first introduced in [141], is the number of complex critical points of the optimization problem (1).

The most basic example is the independence model for two variables having s and t states respectively. Here $n+1 = st$ and $p = (p_{ij})$ is a non-negative $s \times t$-matrix whose entries sum to 1. The set of matrices of rank at most r is the classical determinantal variety. In Sturmfels' work with Hauenstein and Rodriguez [212], the ML degree of X was computed for $s, t \leq 5$. A remarkable duality theorem for these ML degrees was subsequently proved by Draisma and Rodriguez [DR]. Here is the next step:

Problem 1. *Find a formula for the ML degree of the variety of $m \times n$-matrices of rank $\leq r$.*

The algebraic geometry underlying the problem (1) was studied in work of Sturmfels with June Huh [215]. A key object is the *likelihood correspondence* $\mathcal{L}_X$, which is an n-dimensional subvariety of $\mathbf{P}^n \times \mathbf{P}^n$. Points in $\mathcal{L}_X$ correspond to pairs (p, u) where p is any probability distribution that is critical for the likelihood function given by the data vector u as in (1). In this project Sturmfels will address

Problem 2. *Characterize all projective varieties X in $\mathbf{P}^n$ whose likelihood correspondence $\mathcal{L}_X$ is a complete intersection in $\mathbf{P}^n \times \mathbf{P}^n$.*

In the very special when X is a linear subspace, the ML degree of X is a matroid invariant, and, by work of Cohen *et al.* [CDFV], the solution to Problem 2 is given by hyperplane arrangements that are *free* in the sense of Terao; see e.g. [Zie]. Günter M. Ziegler is an expert on the relevant topological combinatorics, and it is planned to work on Problem 2 with members of his group at FU Berlin.

Structural zeros and sampling zeros are an important issue for MLE. In spite of recent progress by Gross and Rodriguez [GR], the following conjecture of Huh and Sturmfels [215, §3] remains open:

Problem 3. *Show that the maximum likelihood degree satisfies the inductive formula*

$$\mathrm{MLdegree}(X) \;=\; \mathrm{MLdegree}(X \cap \{p_n = 0\}) \,+\, \mathrm{MLdegree}(X|_{u_n=0}),$$

provided the variety $X \subset \mathbf{P}^n$ satisfies suitable genericity hypotheses.

Here $\mathrm{MLdegree}(X|_{u_n=0})$ denotes the number of critical points of the likelihood function on the original variety X for data of the form $(u_0, \ldots, u_{n-1}, 0)$ where the u_i are generic. We shall work on this with Klaus Altmann (FU Berlin), using the techniques of T-varieties developed by him and his collaborators.

An important submodel of the low rank model in Problem 1 is the *mixture model*. The points in the mixture model are the $s \times t$-matrices $P = (p_{ij})$ that admit a nonnegative factorization

$$P \;=\; A \cdot \Lambda \cdot B, \tag{2}$$

where A is a nonnegative $s{\times}r$-matrix whose rows sum to 1, Λ is a nonnegative $r{\times}r$ diagonal matrix whose entries sum to 1, and B is a nonnegative $r{\times}t$-matrix whose columns sum to 1. Recall that a matrix P has *nonnegative rank* $\leq r$ if it can be written in the form (2). The study of nonnegative factorizations of matrices and tensors has received a lot of attention recently in engineering and computer science [La]. We plan to attack the following longstanding open problem that dates back to [CR]:

Problem 4. *Given a non-negative matrix with rational entries, does its non-negative rank over the rational numbers agree with its non-negative rank over the real numbers?*

The special case $r = 2$ was resolved in recent work with Kaie Kubjas and Elina Robeva [219, Corollary 3.8]. We are optimistic that an affirmative answer can also be given for rank $r = 3$. Our approach is based on the topological approach developed by Mond *et al.*, and it also involves a careful study of the geometry of the Expectation Maximization (EM) algorithm for maximizing the likelihood function over the mixture model. A fundamental task along the way is to translate the known polyhedral description into a semi-algebraic characterization of the model. This was accomplished for tensors of rank 2 in work with Allman, Rhodes and Zwiernik [211]. The next goal is the extension to tensor rank 3:

Problem 5. *Find explicit quantifier-free semi-algebraic characterizations of the mixtures of three independence models, that is, probability tensors of non-negative rank at most* 3.

A solution to this problem is likely to require the development of a theory that encompasses both the matrix case (where the mixture model fails to be identifiable [219]) and in the general tensor case (where the mixture model is usually identifiable [211]). Its solution would be a significant step forward in the new direction referred to as *semi-algebraic statistics*. The work on Problem 5 will involve a collaboration with the machine learning group of Klaus-Robert Müller at TU Berlin.

Euclidean Distance Optimization

Optimization is a strong direction in the Applied Mathematics landscape in Berlin. Sturmfels will interact with researchers at MATHEON/ECMath and at ZIB Berlin on the following class of optimization problems. Fix a real algebraic variety $X \subset \mathbf{R}^n$, that is, the real solutions to polynomial equations in n unknowns. Given any data point $u \in \mathbf{R}^n$, we wish to compute $u^* \in X$ that minimizes the squared Euclidean distance $d_u(x) = \sum_{i=1}^n (u_i - x_i)^2$ from the given u to X. This optimization problem arises in a wide range of applications, as discussed in [216, 217]. For instance, if u is a noisy sample from X, where the error model is standard Gaussian, then u^* is the maximum likelihood estimate for u.

In order to find u^* algebraically, we consider the set of solutions in $\mathbf{C}^n$ to the equations defining X. In this manner, we regard X as a complex affine variety, and we examine all complex critical points of the squared distance function $d_u(x) = \sum_{i=1}^n (u_i - x_i)^2$ on X. Here we allow only those critical points

x that are non-singular on X. The number of such critical points is constant on a dense open subset of data $u \in \mathbf{R}^n$. That number is called the *Euclidean distance degree* (or ED degree) of the variety X, and denoted as $\mathrm{EDdegree}(X)$. The problem of computing all critical points is referred to as the *ED problem* for X. In light of the remarkable recent advances in numerical algebraic geometry [BHSW], the following will be proposed to one of the two PhD students in this project:

Problem 6. *Develop a numerical homotopy method specifically for the ED problem.*

The geometric theory underlying the ED problem was developed by Draisma *et al.* [216], who found explicit formulas for the ED degree of Segre, Veronese and determinantal varieties. It would be interesting to extend this to other classical varieties such as Grassmannians. This would be useful for approximating antisymmetric tensors with decomposable tensors.

Problem 7. *The Grassmannian $\mathrm{Gr}(d,n)$ of d-dimensional subspaces of an n-dimensional vector space sits in $\mathbf{P}^{\binom{n}{d}-1}$ via the Plücker embedding. Find a formula for its ED degree.*

The starting point for the work on the ED degree was the *triangulation problem in computer vision*, where the relevant variety $X_n \subset \mathbf{R}^n$ is the affine version of the *multiview variety* for n cameras in 3-space as described by Aholt *et al.* [209]. Based on data computed by Stewénius *et al.* [SSN], the following explicit formula for $\mathrm{EDdegree}(X_n)$ was conjectured in [216]:

Problem 8. *Show that the Euclidean distance degree of the affine multiview variety X_n is*

$$\mathrm{EDdegree}(X_n) \quad = \quad \frac{9}{2}n^3 - \frac{21}{2}n^2 + 8n - 4.$$

This connects directly to the ECMath project "Multiview geometry for ophthalmic surgery simulation" led by Michael Joswig (Einstein Professor at TU Berlin). We plan a close collaboration with his group.

A special case of the ED problem is *low-rank approximation* in linear algebra. Given a real $m \times n$-matrix $u = (u_{ij})$, with $m \leq n$, this is the standard optimization problem:

$$\text{Minimize } \; ||\, x - u \,|| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij} - u_{ij})^2} \quad \text{subject to} \quad \mathrm{rank}(X) \leq r. \tag{3}$$

In words: we wish to appropriate u with another matrix x of rank at most r that is closest to u in the Frobenius norm. The solution to (3) is given by the singular value decomposition

$$U \;=\; T_1 \cdot \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m) \cdot T_2,$$

where T_1, T_2 are orthogonal matrices, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ are the singular values of U. By the Eckart-Young Theorem, the closest matrix of rank $\leq r$ to the given matrix u equals

$$u^* \;=\; T_1 \cdot \mathrm{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0) \cdot T_2.$$

In *structured low-rank approximation* [CFP], we are also given a linear subspace $\mathcal{L} \subset \mathbf{R}^{m \times n}$, typically containing the data matrix u, and we wish to solve the following restricted problem:

$$\text{Minimize } \; ||\, x - u \,|| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij} - u_{ij})^2} \quad \text{subject to} \;\; x \in \mathcal{L} \text{ and } \mathrm{rank}(X) \leq r. \tag{4}$$

A best-case scenario is this: if u lies in $\mathcal{L}$ then the SVD solution u^* also lies in $\mathcal{L}$. This happens for some subspaces $\mathcal{L}$, including symmetric matrices and circulant matrices. However, generally things are not this simple. Most subspaces $\mathcal{L} \subset \mathbf{R}^{m \times n}$ of interest do not enjoy this property, and reliably finding the optimal solution to (4) is difficult. First steps were taken in a recent article with Giorgio Ottaviani and Pierre-Jean Spaenlehauer [216]. However, many open questions remain. We shall study relevant numerical problems together with Olga Holtz and Volker Mehrmann (TU Berlin).

Problem 9. *Study the problem (4) for both generic and special subspaces $\mathcal{L}$. With the help of the relevant ED degrees, develop specific Gröbner-based solution strategies for (4).*

Specifically, we wish to derive formulas for ED degrees of sections of determinantal varieties in terms of Chern classes in Schubert calculus. We are especially interested in spaces of Hankel matrices that represent low-rank approximations of symmetric tensors. On the applied side, we wish to make further progress on low rank approximation of kurtosis tensors of format $3 \times 3 \times 3 \times 3$ that arise from brain imaging data in neuroscience [217, §4]. Expertise of Gitta Kutyniok (Einstein professor at TU Berlin) on compressed sensing will be very helpful on the computational side of this project.

Euclidean distance problems, such as (4), can also be approached using techniques such as *sum of squares programming* and *Lasserre hierarchy relaxations*. For an introduction see the recent text book by Blekherman *et al.* [BPT]. For instance, if $X \subset \mathbf{R}^n$ is a variety defined by quadratic equations, then, since the objective function is also quadratic, it is natural to consider the second Lasserre relaxation of the ED problem for X. The original problem translates into a semidefinite program of size $n \times n$, subject to rank 1 constraints. In the relaxation, one drops the rank 1 constraints. From the perspective of algebraic geometry, one seeks to optimize a linear function over the second Veronese embedding of X. The following will be pursued in a collaboration with Peter Bürgisser (TU Berlin):

Problem 10. *Examine the geometry of the second Lasserre relaxation of the ED problem for varieties X defined by quadratic equations. When does the ED degree coincide with the degree of the hypersurface projectively dual to the second Veronese embedding of X?*

The degree of the dual hypersurface is relevant here because it measures the number of critical points for optimizing a *generic* linear function on the given variety [BPT, Chapter 5]. The point of the question is that the linear function derived from squared-distance is quite *special*, and we wish to know whether it nevertheless exhibits the generic behavior. Concerning the relevant vector bundle questions, we anticipate interactions with Dirk Kreimer (HU Berlin) and Alexander Schmitt (FU Berlin).

Convex Algebraic Geometry

Convex algebraic geometry is a field at the interface of algebraic geometry and convex optimization, aimed at studying convex sets that admit a natural algebraic representation [BPT, HN]. The primary example of such a convex set is the cone of positive semidefinite symmetric $n \times n$-matrices. The intersection of this cone with an affine-linear space is called a *spectrahedron*. The image of a spectrahedron under a linear map is a *spectrahedral shadow*. Spectrahedral shadows are more general than spectrahedra: for instance, the convex dual of a spectrahedron is generally not a spectrahedron but it always a spectrahedral shadow. A principal open question in this field is whether every compact convex semialgebraic set is a spectrahedral shadow. We propose to work on the following variant:

Problem 11. *Which compact convex semialgebraic subsets in $\mathbf{R}^d$ are projections of spectrahedra in $\mathbf{R}^{d+1}$? Give a complete characterization of such convex sets in the plane ($d = 2$).*

The postdoctoral fellow in this Einstein research group will work in Convex Algebraic Geometry, and (s)he will greatly benefit from the Berlin expertise in convexity and optimization. In this regard, it is hoped that Christian Haase and Martin Henk will join FU and TU Berlin, respectively.

A direction related to Problem 11 is the geometric study of three-dimensional spectrahedra

$$\mathcal{S} \;=\; \big\{(x,y,z) \in \mathbf{R}^3 \,:\, xA + yB + zC + D \text{ is positive semidefinite}\big\}$$

where A, B, C, D are real symmetric matrices of format $n \times n$. If $n = 1$ then $\mathcal{S}$ is a halfspace, if $n = 2$ then $\mathcal{S}$ is a quadric cone, and if $n = 3$ then $\mathcal{S}$ is a convex body with either 2 or 4 singular points (nodes) in its boundary. The latter is the familiar *elliptope*. The algebraic boundary of $\mathcal{S}$ is known as a *symmetroid* in classical algebraic geometry. It typically has $\binom{n+1}{3}$ nodes, but these can be complex or lie in $\mathbf{R}^3 \backslash \mathcal{S}$. This raises the following question:

Problem 12. *For which integers $0 \leq a \leq b \leq \binom{n+1}{3}$ do three-dimensional spectrahedra $\mathcal{S}$ exist that have a nodes in their boundary and b real nodes in their symmetroid?*

Degtyarev and Itenberg [DI] solved this problem for $n = 4$. There are 20 possible pairs of integers (a, b), namely those where $b \geq 2$ and both a and b are even. Their proof is extremely indirect: it rests on the Global Torelli Theorem for K3 surfaces, and on deep topological results of Kharlamov and Nikhulin. In a recent collaboration with John Christian Ottem, Kristian Ranestad and Cynthia Vinzant [218], we found a new, elementary proof of the Degtyarev-Itenberg result. This allowed us to explicitly constructed all 20 combinatorial types of general quartic spectrahedra.

Quartic symmetroids also play a key role in June Huh's remarkable counterexample to the geometric Chevalley-Warning conjecture [Huh]. For these reasons, it would be very interesting to better understand the codimension 10 subvariety of $\mathbf{P}^{34}$ that parameterizes the variety of all quartic symmetroids. We hope to make some progress on the following difficult problem in enumerative geometry:

Problem 13. *Find the degree and some equations for the variety of quartic symmetroids.*

One of the cornerstones of convex algebraic geometry is the application of semidefinite programming to sum of squares problems. In this application one optimizes linear functionals over the following special class of spectrahedra. Let $f \in S^{2d}\mathbf{R}^n$ be a homogeneous polynomial of degree $2d$ in n variables. Its *Gram spectrahedron* $\mathrm{Gr}(f)$ is, roughly speaking, the set of all representations of f as a sum of square of real polynomials of degree d. More formally, we consider the linear map $\phi : S^2(S^d\mathbf{R}^n) \to S^{2d}\mathbf{R}^n$ that maps quadratic forms in $S^d\mathbf{R}^n$ to forms of degree $2d$. The fiber $\phi^{-1}(f)$ is an affine-linear space of dimension $\binom{\binom{n+d+1}{d}+1}{2} - \binom{n+2d-1}{2d}$. The Gram spectrahedron $\mathrm{Gr}(f)$ is the set of positive semidefinite quadratic forms in that space $\phi^{-1}(f)$. For instance, if $n = 2$ and $d = 3$ then $\mathrm{Gr}(f)$ is a 3-dimensional spectrahedron, whose algebraic boundary is a quartic Kummer surface [218, §5]. For ternary quartics, then $n = 3$ and $d = 2$, the Gram spectrahedron is 6-dimensional [186, §6]. A collaboration between Sturmfels and Raman Sanyal (FU Berlin) will be aimed at resolving

Problem 14. *Study the facial structure of Gram spectrahedra* $\mathrm{Gr}(f)$*. Determine the algebraic degree of semidefinite programming when restricted to the class of Gram spectrahedra.*

The second sentence refers to the results with Nie and Ranestad on general semidefinite programming [177]. The question is to transfer that approach to the more specific case of SOS programming. For instance, for 6-dimensional spectrahedra of degree 6, the algebraic degrees are 112, 1400 and 32. But, these numbers drop to 63, 36 and 1 for Gram spectrahedra [186, §6]. This calls for an explanation.

Our last problem in this section concerns convex polynomials in $S^{2d}\mathbf{R}^n$, a topic of considerable interest in optimization [AP]. A polynomial f is *convex* if its Hessian matrix $H(x) = \big(\partial^2 f/\partial x_i \partial x_j\big)$ is positive semidefinite at every point in $\mathbf{R}^d$. This is equivalent to saying that the bihomogeneous polynomial

$$y^T H(x) y \;=\; \sum_{i=1}^{d}\sum_{j=1}^{d} y_i y_j \frac{\partial^2 f}{\partial x_i \partial x_j} \qquad (5)$$

is non-negative on $\mathbf{R}^d \times \mathbf{R}^n$. The set of convex forms is a convex cone that is contained in the cone of non-negative polynomials in $S^{2d}\mathbf{R}^d$. This containment is strict whenever $d \geq 2$ even for binary forms ($n = 2$). The following problem will be addressed by the other Berlin PhD student:

Problem 15. *Study the boundary of the cone of convex polynomials in* $S^{2d}\mathbf{R}^n$*, starting with* $n = 2$*. In particular, what is the squarefree polynomial that vanishes on this boundary?*

As an illustration, we present the answer to the last question for ternary sextics

$$ax^6 + bx^5y + cx^4y^2 + dx^3y^3 + ex^2y^4 + fxy^5 + gy^6. \qquad (6)$$

Here $d = 3, n = 2$. We seek to characterize those coefficient vectors (a, b, c, d, e, f, g) for which the function $\mathbf{R}^2 \to \mathbf{R}$ is convex. The defines a convex cone in $\mathbf{R}^7$. The boundary of our cone is an irreducible hypersurface of degree 18. Its defining polynomial has 3140 terms:

$$2^{22}3^{12}5^8 \cdot a^6c^3e^3g^6 - 2^{20}3^{12}5^9 \cdot a^6c^3e^2f^2g^5 + \cdots - 3^{24} \cdot a^2d^{14}g^2 + \cdots + 2^{22}5^2 \cdot c^9e^9.$$

This is the largest factor in the discriminant of the bihomogeneous form (5) derived from (6).

Classical Moduli Spaces

Algebraic geometry is the study of zero sets of polynomials. Solutions that depend on an infinitesimal parameter can be analyzed combinatorially using min-plus algebra. This insight led to the development of *tropical geometry*. For an introduction see the text book of Maclagan and Sturmfels [MS]. This is a subject of close collaboration (cf. [158], [167]) with Michael Joswig (Einstein Professor at TU Berlin).

While all algebraic varieties and their tropicalizations may be explored at various level of granularity, varieties that serve as moduli spaces are usually studied at the highest level of abstraction. On the tropical side, this perspective was developed by Mikhalkin [Mik] and others. Sturmfels' recent papers in tropical geometry [210, 214, 220] did exactly the opposite: they explored and tropicalized certain concrete moduli spaces, mostly from the 19th century repertoire, by means of their defining polynomials. A key ingredient was the parametric representation of such moduli spaces by maps whose coordinates are products of linear forms. For the most interesting cases studied in the recent literature, these linear forms arise from the real root systems of type E_6 and E_7, and the complex root system of type G_{32}.

The following problem arose from the work of Hacking *et al.* [HKT] on moduli spaces of marked del Pezzo surfaces. We intend to complete this work in a collaboration with Qingchun Ren and Kristin Shaw, who is planning to move to Berlin. We hope to hire Kristin as a postdoc in the Einstein group.

Problem 16. *Find all combinatorial types of tropical cubic surfaces over the Naruki fan.*

The *Naruki fan* is a 4-dimensional fan with 76 rays and 1215 maximal cones, derived from the Bergman fan of type E_6, which serves as the tropical moduli space of six points in the plane. It is the image of the 6-dimensional *Sekiguchi fan*, which is derived from the Bergman fan of type E_7. This parametrizes configurations of seven points in the plane [214, §6]. The fibers of this map are our *tropical cubic surfaces*. Problem 16 asks for a combinatorial description of these surfaces.

In collaboration with Gavril Farkas (HU Berlin), we shall attack the following related question:

Problem 17. *Determine the prime ideal of the universal Kummer threefold in* $\mathbf{P}^7 \times \mathbf{P}^7$.

Kummer varieties are quotients of abelian varieties by their involution $x \mapsto -x$. Each Kummer threefold has a natural embedding into $\mathbf{P}^7$ by second order theta functions. The moduli space is recorded in a second copy of $\mathbf{P}^7$ by the corresponding theta constants. By taking the closure in the product space, one obtains the 9-dimensional universal Kummer threefold, and the problem is to find its prime ideal in the polynomial ring in 18 unknowns. An explicit set of 891 minimal generators was found in work with Ren, Sam, and Schrader [220], but van Geemen [VG] showed recently that our list was not complete. Using his new approach, we now hope to resolve Problem 17 for good.

The work described above required a substantial amount of numerical computations with Riemann theta functions, using the Sage implementation by Swierczewski [SD]. Alexander Bobenko (TU Berlin) is an expert on computing with theta functions; see [DHBHS] . From a tropical perspective, it is also natural to use non-archimedean versions of theta functions, such as *p-adic theta functions*. In both situations, computations are based on numerical approximations, and it is essential to get a good understanding of error estimates and convergence issues. We shall work on the following task via

Problem 18. *Develop an integrated approach for numerical evaluation of theta functions that links the classical complex floating point setting with the non-archimedian setting.*

A Mumford curve is a curve over a non-archimeadean field K whose Berkovich skeleton has the maximal number of cycles. Every such curve is uniformized by a Schottky group. The algorithms of Morrison and Ren [MR] take the Schottky group as their input. The output is the Berkovich skeleton [BPR] and a canonical embedding of the curve. In this project we will attack the converse problem, which is harder: Given a plane curve $\mathcal{C}$ over K, decide whether $\mathcal{C}$ is a Mumford curve, and, if yes, compute a faithful tropicalization and the Schottky group of $\mathcal{C}$. A complete solution to this problem for cubic curves was given in work with Chan [205]. Building also on [186], the next step is quartics:

Problem 19. *Consider a plane quartic curve $\mathcal{C}$, defined over a non-archimedian field such $\mathbf{Q}$ with the p-adic valuation. Find a practical method to decide whether it is a Mumford curve, and, if yes, compute a tropicalization that is faithful (in the sense of Baker-Payne-Rabinoff [BPR]).*

Our final problem ties back in with quartic spectrahedra and the work of Degtyarev and Itenberg [DI] mentioned in Problem 13. It rests on the global Torelli map for K3 surfaces of degree 4; see [PS]. The period domain for such K3 surfaces is a 19-dimensional analytic manifold sitting inside $\mathbf{P}^{21}$. Each quartic surface $\mathcal{Q} \subset \mathbf{P}^3$ determines a point on that manifold by integrating the nowhere vanishing holomorphic 2-form against 22 linearly independent 2-cycles on $\mathcal{Q}$. Our goal is to turn the global Torelli theorem into an algorithmic tool. The Berlin expertise in numerical methods for topological combinatorics and discrete differential geometry will be very helpful for carrying out this project.

Problem 20. *Develop and implement a numerical method whose input is the equation of a quartic $\mathcal{Q}$ in $\mathbf{P}^3$, and whose output is the corresponding point in the period domain in $\mathbf{P}^{21}$.*

This problem is part of a broader vision that the study of moduli spaces should be integrated into numerical algebraic geometry [BHSW]. It will be advantageous to consider both complex and p-adic numbers at the same time, in the trans-archimedean spirit of Problem 18. For plane curves, the Torelli map has been implemented numerically in `Maple` and `Sage`, using the algorithms of Deconinck *et al.* [DH, DHBHS]. Here, the input is a smooth curve of degree d, and the output is a complex symmetric $g \times g$-matrix whose imaginary part is negative definite, for $g = \binom{d-1}{2}$, i.e. a point in the Siegel upper half space [210]. Problem 20 asks for an extension of this method to quartic surfaces in $\mathbf{P}^3$.

2. Erläuterung der Intensität bereits bestehender bzw. abgeschlossener Forschungskooperationen mit dem vorgeschlagenen Einstein Visiting Fellow

There is probably no high-ranking international research mathematician who has stronger, more diverse, and more productive research contacts and collaborations with researchers and groups in Berlin than Bernd Sturmfels: The present proposal thus has an excellent basis for intensifying and enhancing research cooperations by getting Sturmfels to Berlin, with his own Berlin group, as an Einstein Visiting Professor.

Indeed, Sturmfels was a recipient of the *Alexander von Humboldt Research Prize* 2007-2008 and spent the academic year at TU Berlin. In 2011 Sturmfels was a MATHEON Visiting Professor at FU Berlin. Sturmfels has been invited to speak twice at the "BMS Friday Colloquium" of the Berlin Mathematical School, in 2007 and 2011, and his colloquium talk at the SFB/Transregio "Discretization in Geometry and Dynamics" in 2013 was broadcast live from Berlin to Munich.

Strong ties with Günter M. Ziegler (FU Berlin) and his group have been established over the last 25 years; they resulted in foundational results in the area of discrete geometry and oriented matroids. In particular, the 1992/1999 joint book on oriented matroids is a standard reference in the field. The collaboration was reinforced when Ziegler was a Miller Visiting Professor at UC Berkeley in 2001.

Recent joint research activities with Michael Joswig (Einstein Professor, TU Berlin) and his group resulted in two papers in tropical geometry. A mutual intersection of interests is the effective and practical computation with objects in discrete, algebraic, and tropical geometry. Starting in June 2014, Joswig will be leading the ECMath project CH3 "Multiview geometry for ophthalmic surgery simulation," for which Sturmfels is registered as a cooperation partner.

Convex algebraic geometry is an emerging field that fosters strong relations between numerical algebraic geometry and convex geometry/optimization. Fertile projects have been pursued with Raman Sanyal (FU Berlin) and his algebraic discrete geometry group. The collaboration was started when 2009-2011 Sanyal was a Miller Research Fellow at UC Berkeley hosted by Bernd Sturmfels. Joint ventures so far have led to two well-cited publications.

The Berkeley–Berlin connection will be further cultivated through interaction with Christian Haase, who had been a postdoc at Berkeley and who has been offered a faculty position at Free University. In addition, there are substantial thematic overlaps with the groups of Klaus Altmann (FU Berlin, algebraic geometry), Peter Bürgisser (TU Berlin, algebraic complexity, algorithmic algebra), Holtz/Mehrmann (TU Berlin, numerical linear algebra), and Gavril Farkas (HU Berlin, moduli spaces), Klaus-Robert Müller (TU Berlin, machine learning, algebraic statistics).

3. Stand der Forschung

Nonlinear algebra is a thriving research area at the interface of pure and applied mathematics. In Section 1, we presented four specific research directions, namely Maximum Likelihood Geometry, Euclidean Distance Optimization, Convex Algebraic Geometry, and Classical Moduli Spaces. The following references were cited, and give a sample of the advances and opportunities in these areas:

[AP] A. Ahmadi and P. Parrilo: A convex polynomial that is not sos-convex, *Math. Program.* **135** (2012) Ser. A, 275–292.

[BPR] M. Baker, S. Payne and J. Rabinoff: Nonarchimedean geometry, tropicalization, and metrics on curves, `arXiv:1104.0320`.

[BHSW] D. Bates, J. Hauenstein, A. Sommese and C. Wampler: *Numerically Solving Polynomial Systems with Bertini*, SIAM, 2013.

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Sturmfels himself has been involved in an intensive research effort in experimental and computational in recent years. His work spans a wide range of topics, in particular in computational algebraic geometry and its applications, and almost all of these projects have been collaborative. In the past four years alone, 37 research papers by Sturmfels were either published or accepted for publication, and these projects involved a total of 49 collaborators. These papers contain significant new results in the areas of algebraic geometry, combinatorics, commutative algebra. computational biology, computer algebra, convexity, information theory, finance, matrix theory, multilinear algebra, numerical methods, tropical geometry, optimization, probability, real algebraic geometry, and statistics. With this preliminary work in place, it can be expected that the proposed Einstein Visiting Professor appointment will have a significant research impact, and very positive effect for mathematical landscape in Berlin and beyond.

Sturmfels has also been very active with regard to educational activities, seminars, and course development, including his work on the widely anticipated first text book on tropical algebraic geometry, written jointly with Diane Maclagan [MS]. Sturmfels co-organized a highly successful program on *Algebraic Geometry with a View towards Applications* at the Mittag-Leffler Institute in Djursholm (Sweden) in 2011. Another big recent success was the creation of the activity group on Algebraic Geometry within SIAM (Society for Industrial and Applied Mathematics).

Since 2010, the following 11 mathematicians obtained their PhD under Sturmfels' supervision, and then went on to successful postdoctoral careers: Dustin Cartwright (Yale), Melody Chan (Harvard), Andrew Critch (MBI Ohio State), Angelica Cueto (Columbia), Alex Fink (Queen Mary, London), Shaowei Lin (A-Star, Singapore), Felipe Rincon (Warwick), Anne Shiu (U Chicago), Ngoc Tran (U Texas), Caroline Uhler (IST Austria), and Cynthia Vinzant (Michigan). Note that six of them are female.

The topics and problems proposed in Section 1 above were selected so as to enhance the synergy with two special research programs to be co-organized by Sturmfels in 2014:

Algorithms and Complexity of Algebraic Geometry, in the fall of 2014 at the new Simons Institute for the Theory of Computing at UC Berkeley. The co-organizers for this special semester are Peter Bürgisser, Joseph M. Landsberg and Ketan Mulmeley. The results of that special semester is likely to lead to subsequent joint research efforts with Bürgiesser's algebraic complexity group at TU Berlin.

Computational Algebraic Geometry in the summer of 2014 at the National Institute for Mathematical Sciences (NIMS) in Deajeon (Korea), organized jointly with Hyungju Park. This culminates in 2015 in the Third SIAM Conference on Applied Algebraic Geometry also to be held at NIMS. Sturmfels maintains close ties to researchers in Korea, China and Japan, and this will strengthen the profile of Berlin Mathematics, and in particularly the visibility of the Berlin Mathematical School, in East Asia.

4. Arbeitsplan

Sturmfels plans to establish a working group in Berlin consisting of two PhD students and one postdoctoral researcher. At this point, the first choice for the postdoctoral position would be Kristin Shaw:

`http://www.math.toronto.edu/shawkm/`

Kristin received her PhD at Geneva with Grigory Mikhalkin and is currently based in Toronto, Canada. The PhD students will be recruited by Sturmfels in the context of the Berlin Mathematical School.

The problems stated in Section 1 serve as points of entry and guidance for the research agenda of the group. Graduate students and postdocs supervised by Sturmfels are always encouraged to develop their