(1) Let \( H \) be a proper subgroup of \( G \). By Lagrange’s Theorem (10.10), the order of \( H \) is a proper factor of \( pq \). If \( |H| = 1 \) then \( H = \{e\} \) which is cyclic. Otherwise, the order of \( H \) is a prime number, and then \( H \) is cyclic because the cyclic subgroup generated by any non-identity element must be equal to \( H \), again by Lagrange’s Theorem.

The ambient group \( G \) need not be abelian. For an example, consider the symmetric group \( G = S_3 \). It is not abelian but its order equals \( pq \) for the primes \( p = 2 \) and \( q = 3 \).

(2) Besides the whole group \( A_4 \) and the trivial subgroup \( \{e\} \), there are precisely 8 other subgroups in the alternating group \( A_4 \). There are three subgroups of order 2, namely \( \langle (12)(34) \rangle \), \( \langle (13)(24) \rangle \), and \( \langle (14)(23) \rangle \). The union of these three subgroups is the unique subgroup of order 4, which is isomorphic to the Klein group \( Z_2 \times Z_2 \). Finally, there are four subgroups of order 3, namely \( \langle (123) \rangle \), \( \langle (124) \rangle \), \( \langle (134) \rangle \), and \( \langle (234) \rangle \). Note that there is no subgroup of order 6, as we know from the failure of the converse of Lagrange’s Theorem. The subgroup diagram of \( A_4 \), and plenty of other useful information, can be found at

http://shell.cas.usf.edu/~wclark/algctlg/small_groups.html

(3) There are 20 distinct homomorphisms from \( Z \) to \( Z_{20} \). To specify such a homomorphism \( \psi \), it is necessary and sufficient to specify the image \( \psi(1) \) of the generator 1 of \( Z \), and each element of \( Z_{20} \) is allowed as the image of 1. The image of \( \psi \) will then be the cyclic subgroup generated by \( \psi(1) \). None of these homomorphisms can be injective because \( Z \) is infinite while \( Z_{20} \) is finite. The number of surjective homomorphisms equals the number of generators of \( Z_{20} \), which is \( \phi(20) = 8 \), the value of the Euler-phi function.

(4) Since \( D_5 \) is not abelian, its commutator subgroup \( C \) is a proper subgroup. Moreover, \( C \) is a normal subgroup (by Theorem 15.20). Since \( D_5 \) has order 10, we conclude that the order of \( C \) equals 1, 2 or 5. But it cannot be 2 because any two reflections of the regular pentagon are conjugate in \( D_5 \), so \( D_5 \) has no normal subgroups of order 2. The order of \( C \) cannot be 1 because the following rotation can be written as a commutator:

\[
(12345) = \sigma \tau \sigma^{-1} \tau^{-1} \quad \text{for} \quad \sigma = (13)(45) \quad \text{and} \quad \sigma = (25)(34).
\]

Thus \( C \) has order 5, and it coincides with the group of all rotations of the pentagon. The quotient group \( D_5/C \) has order 2, so it is isomorphic to \( Z_2 \), the unique group of order 2.

(5) The identity element \((e,e)\) is in \( H \), and \( H \) is closed under products \((g,g)(h,h) = (gh, gh)\) and inverses \((g,g)^{-1} = (g^{-1}, g^{-1})\). So, it is a subgroup, and we get (a). For parts (b) and (c) see the solutions to # 3 in the following midterm exam from Spring 2010:

http://tbp.berkeley.edu/examfiles/math/math113—sp10—mt1—Denis%20Auroux—soln.pdf

Please check out some of the other old exams posted by the Math Dept.