Math 113, **Second Midterm Exam**
Thursday, July 29, 10:15am–11:45am

This exam is open book. You may use the text book (to which you can refer) and your own notes, but no electronic devices. Please write your answers in a blue note book. There are five problems, each worth 10 *points, for a total of* 50 *points. Answers without justification will not receive credit. You may look at your graded exam in class on Monday, August 2.*

(1) Fix the polynomial ring $\mathbf{Z}_3[x]$ in one variable over the field $\mathbf{Z}_3$ with three elements, and consider its Frobenius map
$$\phi : \mathbf{Z}_3[x] \to \mathbf{Z}_3[x]\,,\ f(x) \mapsto f(x)^3.$$
(a) Prove that ϕ a ring homomorphism.
(b) The kernel of ϕ is an ideal in $\mathbf{Z}_3[x]$. Find that ideal.

(2) Consider the quotient ring $R = \mathbf{Z}_3[x]/\langle x^3 + x^2 + 2\rangle$.
(a) Prove or disprove: *The ring R is a field.*
(b) Determine the number of elements in the ring R.

(3) Let $R = \mathbf{R}[x, y]$ be the polynomial ring in two variables x and y over the field $\mathbf{R}$ of real numbers.
(a) Find a non-zero proper ideal in R that is prime but not maximal.
(b) Find an ideal in R that is prime but not principal.

(4) Choose a term ordering for the polynomial ring $\mathbf{R}[x, y]$, compute a Gröbner basis of the ideal $I = \langle x+y, x^2+y^2-1\rangle$, and determine the variety $V(I) \subset \mathbf{R}^2$ of the ideal I.

(5) True or false: *Every algebraic extension E of a field F is a finite extension of F.* Give a proof or counterexample.