SUMS OF $N \times 2$ AMAZONS

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Abstract. Amazons is a board game which has enjoyed some popularity since its appearance on the Internet a few years ago. It can be played on boards of arbitrary size. Like subtraction games studied by Ferguson [1974], Amazons is two-person, perfect-information, and zero-sum. Sums of games such as Amazons offer very interesting examples of "central limit theorems" which turn out to be considerably stronger than those encountered in probability theory.

As an illustration of the power of combinatorial game theory, we offer an analysis of all starting positions of Amazons played on sums of $N \times 2$ rectangles, in which each player has one Amazon in each rectangle. The analysis reveals several common yet important phenomena which many players overlook.

Although we assume no special background, seasoned combinatorial game theorists will find that Amazons provides our motivation for a novel exposition of the subject, starting with "results-oriented" thermography. This approach gets to the main results about hot games much more directly than prior expositions, which begin with Conway's theory of canonical forms and a special emphasis on numbers. From the new perspective, Conway's numbers appear much later as a special case: they are the games whose temperatures are negative.

1. Introduction to Amazons

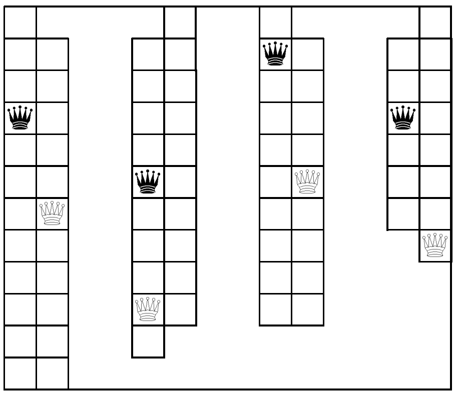

Figure 1. White to move and win. (Blank areas are burned out.)

An Amazon is an immortal chess queen who, after completing a move of her own, then shoots an arrow which also moves like a chess queen. The

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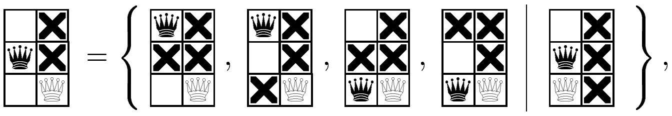
square on which the arrow lands is burned off of the board. Neither Amazons nor arrows may jump over a burned square or over an Amazon. Each player controls several Amazons, exactly one of which must be moved at each turn. The initial board may also contain some squares which have been burned *a priori.* A player who is unable to move loses, and that ends the game.

For readers who enjoy especially challenging problems, we offer Figure 1. A sketch of our solution appears in Appendix B at the end of this paper.

Notation. The study of games such as Amazons has been abstracted, formalized, generalized, and developed by Conway [1973] and by Berlekamp, Conway, and Guy [1982]. This work has grown into the subject now called combinatorial game theory [Nowakowski 1996]. In Conway's now-standard notation, a (two-person, perfect-information, loop-free) *game* (with the last move winning) is defined recursively as two sets of games separated by a vertical bar, known as a *slash:*

$$G = \{G^L | G^R\}.$$

The elements of G^L are the games (i.e., positions) to which Black can move; they are also known as the *Left followers* or *Left options* of G. Likewise, G^R is the set of *Right followers* or *Right options*, the set of positions to which White can move. (L, or Left, is synonymous with bLack, while R, or Right, is identified with whITE.) For example:



where indicates a burned square. In general, each game in G^L and G^R may be similarly defined via further sets.

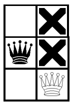
A small set of commonly occurring abstract games have special names, including the game in which neither player can move, so that G^L and G^R are both empty:

$$\{\ |\ \} = 0.$$

Here are two games in which one player can move but the other cannot:

$$\{\ |0\} = +1; \quad \{0|\ \} = -1.$$

In these terms, the current example becomes



$$= \{ \quad -1 \quad , \{0|0\} \ , \quad 0 \quad , \{0|0\} \ \Big| \quad +1 \ \}.$$

To shorten the expressions we often omit braces and use larger and/or multiple slashes to indicate the order of precedence, as

$$-1,\, 0|0,\, 0,\, 0|0 \,\|\, +1\, .$$

Here are some other abbreviations, or special names, for commonly occurring abstract games:

$$\{0|0\} = *\,,\ \text{called "star"};$$
$$\{0|*\} = \uparrow,\ \text{called "up"};$$
$$\{*|0\} = \downarrow,\ \text{called "down"}.$$

If $G = \{A, B, \dots \mid X, Y, \dots\}$ and $G' = \{A', B', \dots \mid X', Y', \dots\}$, the *sum* $G + G$ is defined recursively as

$$\{A{+}G', B{+}G', \dots, G{+}A', G{+}B' \dots \mid X{+}G', Y{+}G', \dots, G{+}X', G{+}Y'\}.$$

That is, a Left move consists in choosing between G and G' and then choosing a Left move in the chosen subgame; and likewise for Right. In the sum of $*$ with another game, the plus sign is often omitted: thus, $1*$ means $1 + *$ and $\uparrow*$ means $\uparrow + *$.

2. Environmental Amazons

Environmental Amazons is a variant of the game which also includes a stack of cards, called the *environment.* Each card represents a number. The stack I have found most useful in classroom demonstrations consists of the following 104 cards: Top card = 5; $9\frac{7}{8}$, $9\frac{5}{8}$, $9\frac{3}{8}$, $9\frac{1}{8}$, $8\frac{7}{8}$, ..., $\frac{1}{8}$, $-\frac{1}{8}$, $-\frac{3}{8}$, $-\frac{5}{8}$, $-\frac{7}{8}$, -1, -1, ..., -1; $-\frac{1}{2}$ = bottom card. This stack is said to have *temperature* $9\frac{7}{8}$ and *decrement* $\frac{1}{4}$. In general, a stack with temperature t and decrement δ begins with a special top card, called the *initial komi*, followed by a card of value t, followed by cards whose values decrease in steps of size δ down to a value in the interval $(-1{+}\delta, -1]$, followed by some number (58, in the present case) of cards of value -1, terminated by an exceptional card of value $-\frac{1}{2}$, which is called the *terminal komi.* Each card has no back; its value is printed on both sides. As in accounting, positive values are printed in black; negative values, in red.

The value of the initial komi is the alternating sum of all the other cards:

$$5 = 9\tfrac{7}{8} - 9\tfrac{5}{8} + 9\tfrac{3}{8} - \cdots + (-\tfrac{5}{8}) - (-\tfrac{7}{8}) + (-1) - \cdots + (-\tfrac{1}{2}).$$

Environmental Amazons rules require the initial komi be given to the *second* player as compensation for the fact that his opponent is allowed to make the first move. If $\mathcal{S}$ denotes the card Stack (excluding the initial komi), then

$$\text{Initial komi} = \mathrm{LP}(\mathcal{S}),$$

where LP denotes the optimal net payoff if Left plays first. If instead Right plays first, we have the Right payoff,

$$\mathrm{RP}(\mathcal{S}) = -\mathrm{LP}(\mathcal{S}).$$

Once the game begins, each player, at his turn, has the option of *either* taking the top card *or* making any legal play on the board. The game continues until the terminal komi on the bottom card is taken; that move is by definition the final move of the game. The position on the board after the terminal komi has been taken is no longer relevant. Each side receives a score equal to the net sum of all of the cards he has taken, and the difference between these scores is the "payoff" which the player with the higher score receives from the player with the lower score. The goal of each player is to maximize the difference between his score and the opponent's score. Tied outcomes are possible, and near-ties (i.e., very small net differences in score) are considered insignificant. A victory by a very small margin results only in a very small payment, without any associated titles, trophies, boasting rights, or other rewards. This is an important difference from popular Amazons, in which the payoff is a step function, allowing the winner to receive the same full prize no matter how close the contest. However, it turns out that it is often much easier to find optimal strategies for playing environmental Amazons, and these strategies often carry over to the popular variant of the game.

Before studying several specific Amazons positions, it is instructive to investigate optimal strategies in the degenerate case in which the game consists entirely of cards. Since each player then has a unique choice for each move, there is only one legal sequence of play, and the result is easily determined: both players receive the same total score, and the net difference between their scores is 0. We can also study modifications of this game obtained by deploying an additional *optional* card of value C, which is placed separately on the table. After the compulsory initial komi, and until the optional card is taken, each player, at his turn, is given the option of taking *either* the card of value C or the top card from the environmental stack. It is easily shown that the optimal strategy for each player is to play exactly as if the card of value C were absorbed into the stack in order. The same result holds if there are several optional cards with assorted values. The best move is always to take the highest available card, whether it resides on the stack or elsewhere.

It is also instructive to compute the resulting scores of games consisting of the specified environment and a few optional cards. The simplest case consists of a single optional card, of value 0. In that case, the optimal net score is also 0. The initial komi cancels out the net effect of all of the other positive cards in the environment, and the terminal komi cancels out the net effect of all of the other negative cards in the environment. More generally, if all optional cards have values which are integers in the range between 10 and -1 inclusive, then a short computation will show that, with optimal play by both players, the net difference between their scores will still be zero. This remains true if the values on the optional cards are allowed to be half-integers or quarter-integers, but it fails if one of the optional cards has value which is not a quarter-integer.

To circumvent this problem, we introduce the *fully enriched environment*, denoted by the symbol $\mathcal{E}_t$. This is an idealization of a very thick stack of cards of temperature t. To specify an "implementation" of this idealized environment for a particular contest, each player bids a proposed value of δ, which can be any positive real number, and a proposed multiplicity of -1s, which must be a positive integer. All bids are then revealed. Ties, if any, may be resolved by coin flipping. The player bidding the smaller multiplicity of -1s then specifies a number larger than his opponent's bid, and the player bidding the higher bound on δ then specifies a positive value of δ smaller than his opponent's bid. This protocol allows either player to ensure any desired (small) upper bound on δ, although his opponent may then try to manipulate within the range between zero and δ. The initial komi of the fully enriched environment is between $t/2$ and $t/2+\delta$, inclusive. The cards occur at small equal decrements ranging from t down to -1. After the specified multiplicity of cards valued at -1, the bottom card on the stack is the terminal komi of value $-\frac{1}{2}$.

For any given specified Amazons board position, a player specified to move "first" (meaning that his opponent receives the initial komi), and a specified value of temperature, t, we might seek expert appraisals of the optimum payoff from experienced gurus. Another guru wishing to challenge an allegedly inaccurate appraisal would do so by specifying the sign of the alleged error and a positive number ϵ smaller than the magnitude of the alleged error. The issue could then be resolved via an ensuing contest. Both contestants would submit bids used to specify an implementation of the environment; the challenger would choose which side he desired to play, and play would commence. In order for the challenger to be successful, he must achieve a net score at least ϵ better than the appraisal.

Since an implementation of a fully enriched environment may contain a very large number or cards, it can be helpful to expedite play by introducing a referee. After both players take the top card on any pair of consecutive moves, the referee interrupts the game and requires each player to submit a sealed bid specifying the temperature (of the top card) at which they next plan to play on the board, assuming that the opponent always takes cards in the meantime. The referee than picks a new temperature equal to the higher bid (possibly incremented by an amount not to exceed 2δ), and removes an *even* number of cards from the top of the stack. In lieu of dealing these cards out appropriately between the two players, the referee ensures the same result by awarding a net score of $(t_{\text{old}} - t_{\text{new}})/2$ to the player who was about to take the top card when the referee intervened.

The only reason for the sealed bids and the referee is to prevent the winning bidder from discovering the size of the losing bidder's bid. A guru confident of his perfect strategy would have no need for such a referee, as he would be willing to let his opponent know his future strategy.

In the jargon of thermodynamics, the presence of a fully enriched environment facilitates an *adiabatic* temperature decrease. Suppose the board game is such that there is some integer, N, which provides an upper bound on the maximum possible number of moves that can be made on the board. Suppose further than either player thinks that perhaps the entire board should be played out at the same temperature. Then he might choose to insist that

$$\delta < \epsilon/N.$$

3. Thermography

A Results-Oriented Definition. A thermograph for a given game position, G, is a pair of functions of the temperature t. One function, called the leftwall and denoted by $\mathrm{LW}(G,t)$, gives the optimal net score for the game $G+\mathcal{E}_t$ if Left plays first; the other function, called the rightwall and written $\mathrm{RW}(G,t)$, gives the optimal net score for the same game if Right plays first. In equations,

$$\mathrm{LW}(G,t) = \mathrm{LP}(G+\mathcal{E}_t) - \mathrm{LP}(\mathcal{E}_t),$$
$$\mathrm{RW}(G,t) = \mathrm{RP}(G+\mathcal{E}_t) - \mathrm{RP}(\mathcal{E}_t),$$

where the value of the initial komi is $\mathrm{LP}(\mathcal{E}_t) = -\mathrm{RP}(\mathcal{E}_t) = t/2$.

By universal convention, thermographs are plotted with the independent variable increasing upward along the vertical axis and the values of the walls as the horizontal coordinate, increasing toward the left rather than the right. Although these conventions differ from those which are common in subjects such as calculus and analytic geometry, they have several desirable properties:

1. The analogy between thermographs and probability distributions is more apparent.
2. The thermographs correspond more directly with the corresponding game graphs, in which play, like temperature, normally progresses in a top-down direction, and Left generally strives to move Leftwards.

Examples. The cardstack $\mathcal{E}_{-1}$ has initial komi of $-\frac{1}{2}$, no matter whether the number of -1s is even or odd. The cards are then -1, -1, $\ldots$, -1; $-\frac{1}{2}$ = terminal komi. Since both the initial and final komis have value $-\frac{1}{2}$, and all non-komi cards are integers, it is clear that for *any* board game, G, the values of $\mathrm{LW}(G,-1)$ and $\mathrm{RW}(G,-1)$ are integers. If $\mathrm{LW}(G,-1)$ and $\mathrm{RW}(G,-1)$ are the *same* integer, k, we say that G is also an integer, k.

If $G=k$, an integer, then for all t, $\mathrm{LW}(G,t) = \mathrm{RW}(G,t) = k$. As long as the environment, $\mathcal{E}_t$, contains cards bigger than -1, both players will prefer to take cards instead of playing in G. However, when the temperature of the environment reaches -1, then the player favored by the sign of k will start using up his moves in G while his opponent is forced to take cards of value -1.

Thermographs of three Amazons positions are shown in Figure 2.

In Figure 2a, Right cannot move, but Left can retreat and then shoot back into his own footprint. The tactic of moving like a chess king and then shooting into her own footprint is called a *plod.* Plodding often turns out to be a good Amazons endgame tactic. In this case it ensures that Left gets

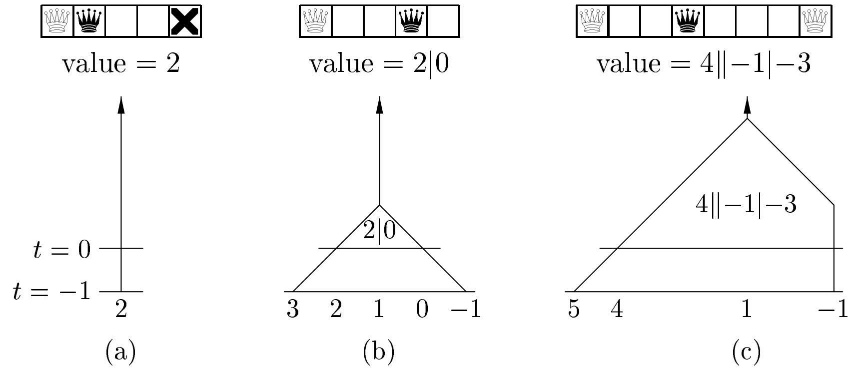


FIGURE 2. Thermographs for some $1 \times N$ Amazons positions.

two consecutive moves while Right gets none. These two moves can be taken when the temperature of the environment is -1, and White gets stuck with two cards of value -1 while Black is able to play two moves on the board. Since neither player will move from this board position until the temperature reaches -1, the position is worth a net score of two points for Left. This and other integer values are independent of the initial temperature, t.

In Figure 2b, each player can move adjacent to the other and then *throw away* by shooting the arrow as far as possible. After a first board move by Left, she has a board position worth two points, but after a first board move by Right, the net board position is worth zero. So 2b behaves as the sum of a 1-point card and a constant 1-point advantage for Left. If the initial temperature exceeds 1, both players will play in the environment until its temperature is played down to 1. At temperatures below 1, both players will play on the board. If the initial temperature is 0, we have

$$\mathrm{LW}(0) = 2, \quad \mathrm{RW}(0) = 0.$$

In 2c, if Left plays first, she will move into contact with one of the opponents and then shoot her arrow onto the square next to the other. If Right moves first, he will lay claim to 2 of the three empty squares on the east side of the board. When the temperature of the environment exceeds 3, both players will play in it. At temperatures between 3 and 1, both players will want to make the first move on this board position, but if Right makes the first move on the board, neither player will want to play on the board again until the temperature falls to 1.

Bottom-up Thermography. The definition of thermographs introduced in the prior section was "results-oriented", or "top-down". A major advantage of this definition is that it is directly applicable to even the more complicated models of loopy positions in Go [Spight 1999]. However, for many calculations, a "bottom-up" approach is more effective.[1] This definition is recursive, working from the leaves of the tree of positions of G back to its root. For this purpose, the values of the leaves may be taken as integers. Before computing the thermograph of G, we assume we already have the thermographs of G^L and G^R.

To build *walls* from the bottom-up, we construct *scaffolds* and *frames*. There are two of each (scaffolds LS and RS, and frames LF and RF), depending on who plays next. Each of them is a function of the temperature, t, and each position of G has a complete set of all four functions. They are defined as follows:

Scaffolds:

$$\mathrm{LS}(G,t) = \max_{G^L} \mathrm{RF}(G^L) - t,$$
$$\mathrm{RS}(G,t) = \min_{G^R} \mathrm{LF}(G^R) + t.$$

Frames:

$$\text{if } \mathrm{LS}(G,t) > \mathrm{RS}(G,t) \text{ define } \mathrm{LF}(G,t) = \mathrm{LS}(G,t), \quad \mathrm{RF}(G,t) = \mathrm{RS}(G,t);$$

$$\text{if } \mathrm{LS}(G,t) \le \mathrm{RS}(G,t) \text{ define } \mathrm{LF}(G,t) = \mathrm{RF}(G,t) = \mathrm{LS}(G,\tau),$$
$$\text{where } \tau \text{ is the smallest real such that } \mathrm{LS}(G,\tau) = \mathrm{RS}(G,\tau).$$

τ is called the *temperature* of G, and $\mathrm{LS}(G,\tau) = \mathrm{RS}(G,\tau)$ is called the *mast* of G.

Theorem: **WALLS = FRAMES.**

An analytic proof is given in Appendix A.

Hotstrat and sentestrat. There are several strategies one might employ which use thermographs to determine where to play on a sum of several board positions plus a fully enriched environment, such as

$$W + X + Y + Z + \mathcal{E}_t.$$

The most naive strategy, called *Hotstrat*, advises one to play on the hottest available position. This strategy turns out to be flawed.

A better strategy, called *Sentestrat*, advises the player to make a local response to his opponent's prior move if that move left a local position there

[1]The bottom-up approach to thermographs requires subtle revisions to handle locally loopy positions such as kos in Go. The situation is very interesting; see [Berlekamp 1996]. The theory partitions kos into two classes, called "placid" kos and "hyperactive" kos. Mast values of hyperactive kos depend on which player has the greater number of kothreats, but mast values of placid kos do not. Even very good Go players have difficulty distinguishing or articulating the difference. The bottom-up theory presented in [Berlekamp 1996] has been very successful at handling sums of regions including at most one hyperactive ko plus any number of placid or kofree regions, but global board positions involving multiple hyperactive kos require top-down techniques [Spight 1999].

whose temperature exceeds t, the temperature of the environment. If not, then the opponent's prior move is not urgent, and the player can revert to hotstrat.

Using sentestrat, it can easily be shown that

$$\mathrm{RW}(G+H,t) \geq \mathrm{RW}(G,t) + \mathrm{RW}(H,t)$$

and, equivalently,

$$\mathrm{LW}(G+H,t) \leq \mathrm{LW}(G,t) + \mathrm{LW}(H,t).$$

This yields the proof of the following

Theorem: **MEANS ADD.**

Subzero Thermography. Figure 3 shows the thermographs of some games whose temperatures are negative, and Figure 4 shows the thermographs of some games whose temperature is zero. All of these are common games with special names.

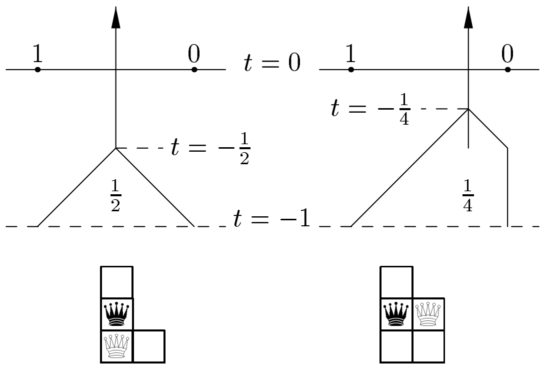


FIGURE 3. Thermographs of some numbers at negative temperatures.

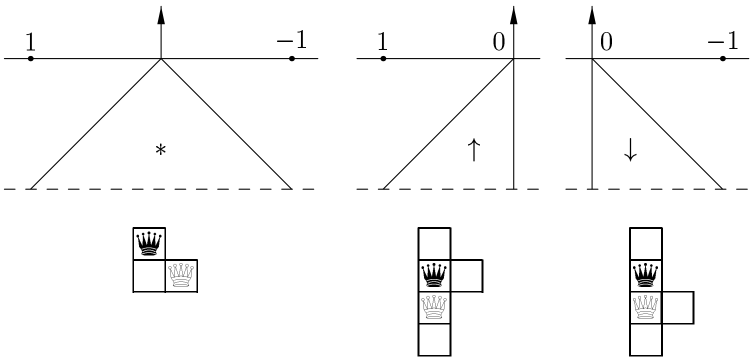


FIGURE 4. Thermographs of some infinitesimals.

From the rules for constructing scaffolds and frames, it is apparent that as temperature decreases, each wall of the thermograph can have only two slopes: vertical, or at 45° away from the vertical in the appropriate direction. If not vertical, the leftwall must slope downward to the left; the rightwall, downward to the right. As temperature moves upward, the left and right walls cannot approach each other at any rate faster than 2 units per degree of temperature. At $t = -1$, all walls are integers. If the walls do not yet coincide, they must be at least one unit apart, so they cannot meet at any temperature between -1 and $-\frac{1}{2}$. Hence, at $t = -\frac{1}{2}$, all walls must be half-integers. Repeating this argument, at $t = -\frac{1}{4}$, all walls must be quarter-integers. Continuing this fractal-like argument, it follows that at $t = -2^{-k}$, the only possible wall values are quantized to integer multiples of 2^{-k}.

Furthermore, if one knows the value of a wall at $t = -2^{-k}$, its downward slope is easily determined. It must be vertical or 45°, accordingly as $\mathrm{Wall}(\,\cdot\,, 2^{-k})$ is an even or odd multiple of 2^{-k}, because this is the only way the wall can meet the quantization constraint at $t = -2^{1-k}$. So evidently,

given a pair of thermographic walls at any negative temperature, there is a unique way to extend them downward to $t = -1$.

In particular, if the base of the mast of a game has negative temperature, we can construct the entire thermograph from the mast value alone. Such a game is called a *number*, and named according to the value of its mast. The games shown in Figure 3, whose trees are $\{0|1\}$ and $\{0\,\|\,0|1\}$, are known as $\frac{1}{2}$ and $\frac{1}{4}$, respectively. It is easily seen that a sum of numbers must be the appropriate number.

However, a downward extension of a wall from $t = 0$ is *not* unique; it can be either vertical or at 45°. But this single bit of information is sufficient to determine the continuation of the unique extension downward to $t = -1$. This bit can have a significant effect on the mean of a game of which it is a position. For example, $5|0\,\|\,1$ and $5|{\downarrow}\,\|\,1$ have mean 0, but $5|{*}\,\|\,1$ and $5|{\uparrow}\,\|\,1$ have mean $\frac{1}{2}$.

All games such as those in Figure 4, having thermographs whose mean and temperature are both 0, are called *infinitesimals*. The number 0 is also considered an infinitesimal. It is easily seen that sums of infinitesimals are infinitesimal. Further discussion of infinitesimals appears in Section 6 (page 18).

Henceforth, we will suppress those portions of thermographs which occur at temperatures significantly below zero. Since a change of slope at $t = 0$ is relatively rare, whenever a wall is discontinued at $t = 0$, the reader should assume an implied continuation of the slope just above 0.

For many purposes, a game of Environmental Amazons can stop when both players are about to begin taking negative cards. If both players have played well, the value of the board position is then necessarily a number, x, which may be interpreted as the “score”. The *leftscore* and *rightscore* of

G are defined as $\mathrm{LW}(G, 0)$ and $\mathrm{RW}(G, 0)$, respectively. Since both players know that competent play for the remainder of the game will result in a net increment of "x" to whatever net cardscore already exists, they might very plausibly agree to view "x" as the "final" score on the board, and halt the game at temperature zero. From this perspective, the *only* purpose of continuing to play at below-zero temperatures would be to resolve the scoring dispute which would arise if the players did not agree on the value of the number x.

4. Central Limit Theorems

Many typical endgames in Amazons or Go can be partitioned into the sum of several disjoint subpositions, each occurring in its own region of the board. Each of these subpositions can be viewed as a separate "game". We are then interested in the sum of these games, which is the whole board. On a very large board, we should expect to find an endgame composed of a large number of summands. Hence, the gamesman should be interested in sums of many games.

As in probability, the mean value of the sum of things is the sum of their mean values. For games, this is the boldfaced theorem on page 9, "means add". But the dispersion about this mean is much more constrained in combinatorial game theory than it is in probability. In probability, the standard deviation of the sum of n objects typically grows as the square root of n. On the other hand, in combinatorial game theory, the dispersion is constrained by an upper bound independent of n.

To see this, one need only observe that the walls of the thermograph of any loopfree game are a continuous concatenation of straight line segments, each of which has a slope that is either vertical or at 45 degrees. It follows that if the base of the mast occurs at temperature t, then both walls of the thermograph must intercept the $t = 0$ axis at a distance no more than t from the mast. Since the sum is itself a game, the same result still applies. The Leftscore cannot exceed the mean by more than t, and the Rightscore cannot be less than the mean minus t. Here t may be taken as the temperature of the hottest summand. So in combinatorial game theory, we have an extraordinarily strong law of large numbers.

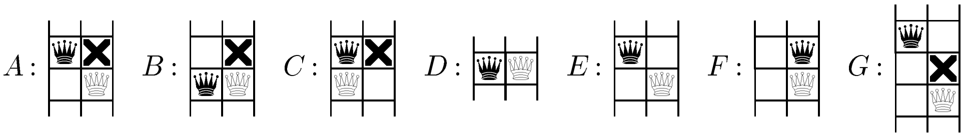
Adding a fully enriched environment of sufficiently high temperature, t, to any sum of games eliminates the dispersion completely. The Leftscore becomes the mean plus $t/2$; the Rightscore becomes the mean minus $t/2$. We view the quantity $t/2$ as the komi given to the second player to balance against the advantage of playing the first real move.

5. The Big Picture of Orthodox Amazon Openings

Moves which determine the walls of the thermographs are called *orthodox* moves. Orthodox moves which determine the masts are especially important.

We are now ready to address a central problem of $N \times 2$ Amazons. If a pair of opposing Amazons occupy arbitrary initial positions in an otherwise empty board, what are their orthodox opening moves? Let x denote the fraction of the total playable area of the initial board (including the squares occupied by the two Amazons) which are south of the center of the row initially occupied by White. Let x' denote the same fraction measured relative to the initial location of Black.

Among the many types of positions which might arise after the first move, we focus our primary attention on those shown here:



Each of these positions, or "games", includes a pair of opposing Amazons, in contact with each other. The diagram may also show an ✖, indicating as usual a burned square near the two Amazons. All other burned squares, if any, are assumed to be adjacent to the top or bottom edges of the board, where they might be accommodated by redefining those edges, which might then become jagged. With each position are associated the following parameters:

k = number of empty squares no higher than ♕

n = number of empty squares no lower than ♛

$m = n - k$

Each of these games has several close relatives. Reflections of the board about the vertical axis yields another games which is equivalent. If we exchange Right and Left (White and Black), we have a similar class of positions which we denote by A', B', ... , G', with parameters k', n', and m'. Finally, turning the games upside down yields Ɐ, ꓭ, Ɔ, ᗡ, Ǝ, Ⅎ, and ⅁. These upside-down letters are used to warn the reader of a significant difference from the right-side up letters. Thus, for example, if Black moves onto the same row as White, she might shoot diagonally either to reach B or to reach ꓭ. Here ꓭ is equivalent to B with different values of k, n, and m.

Since the games A, B, ... , G depend primarily on m, we call m the inner index. In fact, if m is a positive integer, then from either $A[m,n]$ or $C[m,n]$, White's best move is a plodding confrontation, and Black's best move is a plodding retreat. Neither player will want to make such a move at any temperature above -1, and so for any n, $A[m,n] = C[m,n] = m$.

Opposite files. We first consider the case in which the Amazons start on opposite files. Intuitively, if $x < \frac{1}{2} - \frac{1}{N}$, Black's opening move to A looks very strong. It concedes the bottom portion of the board to White, but reserves the larger portion at the top of the board for Black. If instead $x > \frac{1}{2} + \frac{1}{N}$, then Black can play a centrally symmetric move which yields an upside-down version of A. This move looks so very strong that there is a strong temptation to conjecture that, at least for purposes of computing its mean value, we might accurately estimate the initial position S as

$$S \sim A, \forall | A', \forall' = m | {-m'} \text{ if } \left| x - \tfrac{1}{2} \right| > \tfrac{1}{N} \text{ and } \left| x' - \tfrac{1}{2} \right| > \tfrac{1}{N}.$$

A closer investigation uncovers a relatively minor exception: if $|x - x'| \leq \frac{1}{N}$, then possibly Black's Amazon initially occupies the square to which it desires to move. However, in this case, Black can plod to C. This leads to the refined conjecture,

$$\text{If } \left| x - \tfrac{1}{2} \right| > \tfrac{1}{N} \text{ and } \left| x' - \tfrac{1}{2} \right| > \tfrac{1}{N}, \text{ then } S \sim A, \forall, C | A', \forall', C' = m | {-m'}.$$

In the limit as N goes to infinity, this conjecture turns out to be true in only $\frac{102}{121}$ of the unit x, x' square. The reason is that within a region of area $\frac{19}{242}$, Left has another move that yields a full point improvement over both A and $\forall$, and there is a complementary region wherein Right can do likewise. Figure 5 reveals the awesome truth. It shows Left's orthodox opening move everywhere within the unit x, x' square except in the very close vicinity of the center, a singular point whose detailed study is deferred until subsequent sections. When the Amazons initially occupy the same file, Black's orthodox opening is shown in Figure 6.

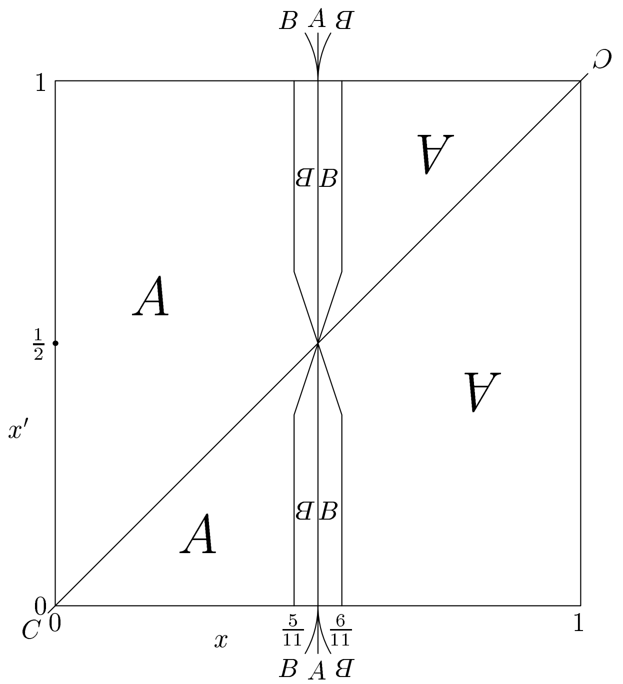


FIGURE 5. Black's orthodox opening move when Amazons start on opposite files.

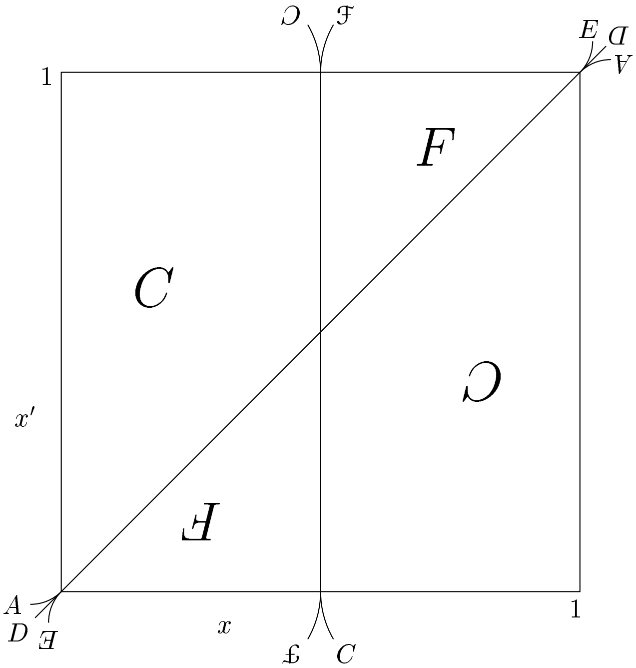
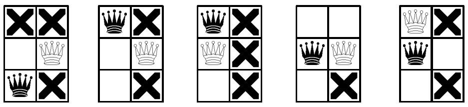


FIGURE 6. Black's orthodox opening move when Amazons start on the same file.

When the Amazons initially occupy different files, a surprising position that often proves superior to $A[m,\infty]$ is Ǝ, a value equal to $B[-m-4,\infty]$:

$$B[-m-4,\, n] = \quad n-2 \quad\mid\quad m+2 \quad\|\quad -m-1$$



With $n=\infty$, its thermograph appears as the solid picture in Figure 7, superimposed on the dashed thermograph of $A[m,\infty]$. At temperatures below $2m+1$, A is the better left option from S, but at higher temperatures, B is better than A if n is sufficiently large. Figure 7 is used to construct the left scaffold of an approximation to the starting position S, as shown in Figure 8.

So two conditions need to be met in order for the mast of S to be determined by option B rather than by option A:

(1) n must be sufficiently large to ensure that $B[-m-4,\, n]$ behaves like $B[-m-4,\infty]$.
(2) The right scaffold of S must be sufficiently far away that the temperature of S is not less than $2m+1$.

Condition 1: The relevant expression for $B[-m-4,\, n]$ is

$$n-2|m+2\,\|\,-m-1,$$

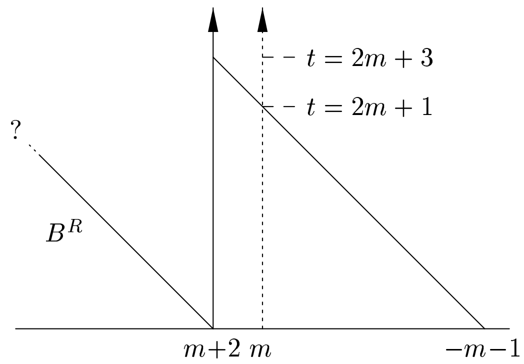


FIGURE 7. $A = m$ (dotted) and $B = \infty|m{+}2 \,\|\, {-m}{-}1$.

whose temperature first exceeds $2m+1$ precisely when the temperature of $n-2 \,|\, m+2$ exceeds $2m-1$. This occurs when and only when

$$n - 2 > m + 2 + 2(2m-1),$$

or

$$n > 5m + 2.$$

The numbers of squares above and below-or-equal the white Amazon is

$$\begin{aligned} a &= \#\,\text{above} &&= n+1, \\ b &= \#\,\text{below-or-equal} &&= n+m+2, \end{aligned}$$

whence the condition we seek can be restated as

$$6a > 5b - 2,$$

or, in words,

> Black's Option B is orthodox only if the White Amazon is located strictly within the central *eleventh* of the board.

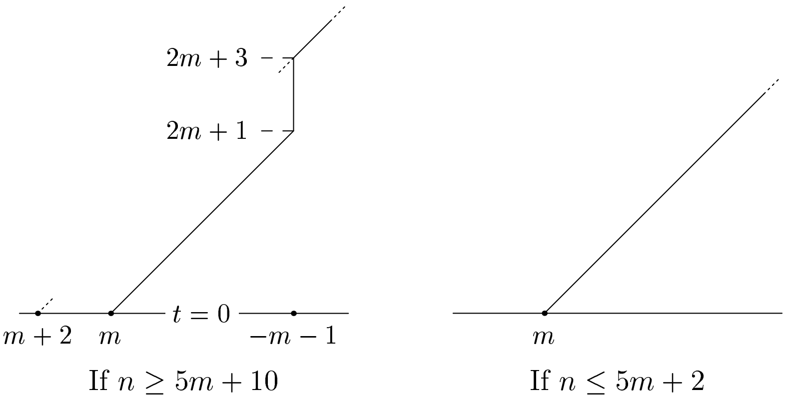


FIGURE 8. Left scaffold of $S = \{m,\ n{-}2|m{+}2 \,\|\, {-m}{-}1 \;|||\; S^R\}$.

(To make this assertion precise, "strict" should be interpreted in accordance with the prior inequality.)

If the White queen is not located strictly within the central eleventh of the board, then Black's plan to trap her in three consecutive moves is unorthodox because after White is trapped, Black owns only the smaller side of the board, which isn't a large enough payoff to compete successfully with Black's straightforward option A.

Condition 2: In order for option B to prevail, it is also necessary for the temperature of S to be at least $2m + 1$, and this imposes a constraint on S^R, which depends only on the starting location of the Black Amazon. By symmetry between Black and White, the right options of S include $-m'$ and $m'{+}1\|m'{-}2|{-}n'{+}2$. The desired high temperature of S occurs only when $m' - (m + 1) > 2m + 1$, or $m' + 1 > 3m + 3$. Since the magnitude, in quarter-rows, of the distance between the White Amazon and the centerline of the board is precisely $m + 3$, we may restate this condition as follows:

> Black's option B is orthodox only if the row above the Black Amazon is more than 3 times as far away from the center of the board as White's.

Interpretation: When the values of x and x' lie in the appropriate regions of Figure 5, Black's orthodox opening move goes onto the same row as white and then places an arrow directly *behind* her, impeding her retreat from the center.

If White plays elsewhere, the purpose of Black's opening move becomes apparent when Black completes his second move. This threatens a third Black move which would *trap* White in a region with only one empty square, while Black claims nearly half of the original board. In the typical cases, this threat is sufficiently large that White must respond, after which White has avoided the trap, but Black is now two points better off than if he had played option A originally. In other words:

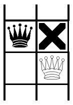
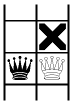
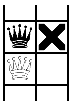
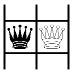
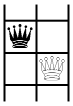
Option A (block center):	move sequence = L;	result is $+m$
Option B (impede retreat):	move sequence = LLR;	result is $+m + 2$

So option B becomes the orthodox choice whenever the temperatures of the relevant positions have the proper order, and this occurs precisely when the values of x and x' lie in the appropriate regions of Figure 5.

Toss-aways. In Figure 6, Black's orthodox opening to D, E, or F entails a throw-away, which shoots the arrow against a distant edge of the board. One might instead play a *toss-away*, in which the arrow lands short of the board's edge. The only case in which this yields an improvement is $F[-3, v]$, and then only when the jaggedness is favorable and the arrow lands one square short of the edge. We denote this position by $\mathcal{F}$. It will be illustrated in Appendix B.

TABLE 1. Values of nearly centered, sparse, $N \times 2$ Amazon positions

k = number of empty squares no higher than ♕
n = number of empty squares no lower than ♛
$m = n - k$
u = *any* infinitesimal game for which $0 < u{\downarrow} << {\uparrow}$

m	-1		0		1		2	
n	v	d	v	d	v	d	v	d
A: 	$\frac{1}{2}\|0$ $A_8{=}*$ $A_4{=}*$	$u*$	0	$\frac{1}{2}$ $A_9{=}0$ $A_5{=}0$	1	1	2	2
B: 	$u*$	$*$ $B_7{=}u*$ $B_3{=}0$ $B_1{=}\frac{1}{4}$	$\frac{1}{2}$ $B_8{=}0$ $B_4{=}0$	0	$\frac{1}{2}\|0$ $B_{10}{=}*$ $B_6{=}*$	$*$ $B_9{=}u*$	$\supseteq 1\|-1$	$\supseteq 1\|-1$
C: 	-1 $C_2{=}*$ $C_0{=}*$	-1 $C_3{=}-\frac{1}{2}$ $C_1{=}{\downarrow}$	0	0 $C_3{=}\frac{1}{2}$ $C_1{=}\frac{1}{2}$	1	1	2	2
D: 	$*$	$*$ $D_3{=}\frac{1}{4}$ or $-\frac{1}{4}$ D_1 is B_1 or $-B_1$	0	0	$*$ $D_4{=}\frac{1}{4}$ or $-\frac{1}{4}$ D_2 is B_1 or $-B_1$	$*$	$1\|-1$ $D_4{=}0$ $D_2{=}0$	$1\|-1$ $D_5{=}\frac{1}{2}\|-\frac{1}{2}$ $D_3{=}*$
E: 	$\frac{1}{2}\|0$ $E_8{=}*$ $E_4{=}*$	$*$	$1\|-1$ $E_4{=}\frac{1}{2}\|-\frac{1}{2}$ $E_2{=}*,{\uparrow}\|*,{\downarrow}$ or or…	$1\|-1$ $E_3{=}0$ $E_1{=}0$	$*$	$0\|-\frac{1}{2}$ $E_9{=}*$ $E_5{=}*$	$1\|-u*$	$1\|*$ $E_9{=}1\|-u*$ $E_5{=}1\|0$ $E_3{=}1\|-\frac{1}{4}$
	$\not\geq 0$ unless $2n-m=7$ or 3		$\not\geq *$ except A_d, B_v, C		$\not\geq 0$ except A, C, or unless $2n-m=7$ or 3		$\not\geq *$ except A, C, and E_5	

6. Central Amazonian Details

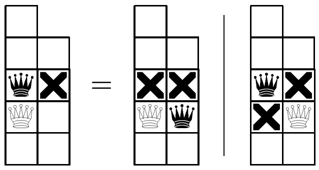
Table 1 provides an overview of the values of five classes of $N \times 2$ Amazon positions, or games. The table considers only positions for which $m = -1$, 0, 1, or 2. The parameter n may be arbitrarily large, but the Amazons must both be very near the center of the board. The results depend primarily on m, the inner index, and also somewhat on the parity of n, which is denoted by the letter v or d, for eVen or oDd. The boldface entries shown in the table apply to all sufficiently large n of the appropriate parity. All exceptions for small n are specified explicitly. Since the value of m is constant in each column, only the n subscript is included for each exceptional case.

Although some entries have novel meanings to be explained below, the table also includes many familiar games such as 0, $\frac{1}{2}$, and $*$. Each entry labeled $u*$ has the same thermograph as $*$. Except for the distinction between $u*$ and $*$, some properties of $B[2,\cdot]$, and $E[0,2]$, our primary emphasis is still on thermographs and means.

Many Amazons players are surprised by the fact that the value $\frac{1}{2}$ continues to persist even for very large n. The reasons for this, which we will soon explain, are subtle.

This table was generated by a diligent "top-down" approach. Rather than working upwards from positions with smaller numbers of squares to positions on increasingly large boards, in many instances it proved easier to analyze the large boards first. The major difficulty of the "bottom-up" approach to these positions is that many of the games classified in Table 1 can lead to more complicated positions which are much less well understood.

Most of the Amazons positions categorized in Table 1 have many followers. But typically there are only a very few orthodox followers. For example,

$C[0,3] =$  $= 0|1 = \frac{1}{2}.$

Depending on the parities of n and m, the edges of the board may be either smooth or jagged. The top edge of the version of $C[0,3]$ shown here in the text is jagged but its bottom edge is smooth. When an edge is jagged, it is conceivable that the value of the position depends on whether it is left-jagged or right-jagged. However, in the case of $C[0,3]$, it is easily verified that the direction of jaggedness does not affect the game. In fact, this is nearly always the case. The only exceptions are $D[-1,3]$ (which is an upside-down version of the identically valued $D[1,4]$), $D[-1,1]$ (upside-down version of $D[1,2]$), and $E[0,2]$, whose four variants are all distinct.

Invasions which beat 0 to gain $\frac{1}{2}$ point. We use the term *gain* to refer to the difference between the actual value, which includes the invading move, and the hypothetical *default* value the game would have if the invading option were disallowed, or if it were not worthwhile. (Be careful not to confuse these hypothetical "gains" with the "incentives" that appear later in this paper. Left's half-point *gain* results from the existence of a worthwhile option. His *incentive* to play it is actually $-\frac{1}{2}$.)

We have already stated that $A[m,n] = C[m,n] = m$ for $m \geq 1$. In all such cases, Black can do no better than a plodding retreat. At first, one might mistakenly think that a similar argument ought to work when $m = 0$. For $C[0,n]$, it does work for all values of n except 3 and 1. It is only in those two cases that Left has a worthwhile invading option which gains $\frac{1}{2}$ point.

The general case depends critically on subzero thermography.

Given a game G, we say that $G \lesssim 0$ if and only if there exists some negative temperature, t, such that $\mathrm{LW}(G,t) \leq 0$. Similarly, we say that $G \gtrsim 0$ if and only if there exists some negative temperature, t, such that $\mathrm{RW}(G,t) \geq 0$. Since walls are monotonic functions of t, it is apparent that the most promising negative values of t are those very close to zero. Since $\mathrm{RW}(G+H,t) \geq \mathrm{RW}(G,t) + \mathrm{RW}(H,t)$, it is apparent that the sum of two games that are $\gtrsim 0$ is also $\gtrsim 0$.

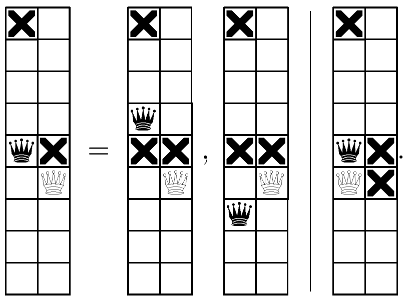
The above formula for $C[0,3]$ is a special case of the general formula for $A[0,n]$ and $B[0,n]$:

$$\{-1, V | {+1}\} = \begin{cases} 0 & \text{if no } V \gtrsim 0, \\ \frac{1}{2} & \text{if some } V \gtrsim 0 \text{ but no } V \gtrsim \frac{1}{2}, \end{cases}$$

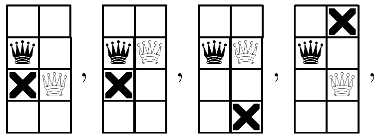
where V is the set of Left's invading moves, and where -1 and $+1$ are the values of the positions reached respectively by Left's plodding retreat and Right's plodding confrontation. The relevant temperature range of the thermograph of V is near $t = 0$ and/or $t = -\frac{1}{2}$.

We first show that any worthwhile invading move played from $A[0,n]$ must shoot back onto the square from which Left came, thereby sealing off the top half of the board. If instead Left completed her turn in any way which left that square empty, then Right could win by plodding there. So if Left has a worthwhile invasion, it must reduce the size of the board in play. After Left's invading move, Right plays next on a new, initially empty board on which the Amazons occupy opposite files. However, the relevant temperature range is much lower than in the assumptions underlying Figure 5.

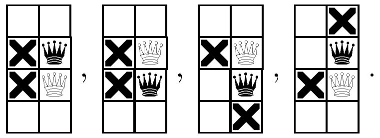
For example, consider

$A[0,7] =$ 

In general the two Left moves shown here dominate all others. To show that the invading move is worthwhile, we must show that it has no Right follower ≤ 0. The four most promising candidates are

$V^R \quad =$ 

but each of these can be thwarted by Left's move to the position directly below:

$V^{RL} \quad =$ 

In order to compare these positions with those listed in Table 1, it is helpful to negate them by swapping the locations of the two Amazons. The positions in this set then become

$$-V^R = A[-1,2],\ B[1,3],\ D[1,3],\ E[1,3].$$

We could also reach this set of positions by negating the original A or B position, and considering Right invasions from $-A[0,7]$. Thus, the assertion that there is a worthwhile invading move from $A[0,7]$ (or equivalently, from its negative) is equivalent to the assertion that the following positions are all $\ngtr 0$:

$$A[-1,2],\ B[1,3],\ D[1,3],\ E[1,3].$$

The bounds shown at the bottom of Table 1 reveal that, except for certain very specific small n, all other tabulated values with odd m are $\ngtr 0$. The diligent reader can then verify that this implies that worthwhile invading moves do indeed exist from $A[0,d]$ and from $B[0,v]$, unless $\lfloor n/2 \rfloor = 4$ or 2.

In general, an invasion from A, B, $-A$, or $-B$ cannot be worthwhile unless the invader jumps to a square fairly close to the center of the lower half of the board. If the invader lands too far from the center, the defender can respond with a move to a position equivalent to type A or $-A$ (possibly after reflection about some vertical or horizontal axis, and possibly after changing sign by swapping the two Amazons), in which $|m| \geq 2$ and the defender has achieved a very desirable value by blocking the invader away from the center. So the assiduous reader can show that the invasion locations which merit the most consideration are those for which the defensive response yields a position equivalent to one of those shown in Figure 1, with $m = -1$, 0, 1, or 2.

Amazonian thermographs. In our earlier discussion of Figure 5, we noted the existence of an exceptional line at $x = \frac{1}{2}$. The array of thermographs presented in Figure 9 provide the basis for further study of these exceptional lines. Each column of Figure 9 shows the thermographs of all

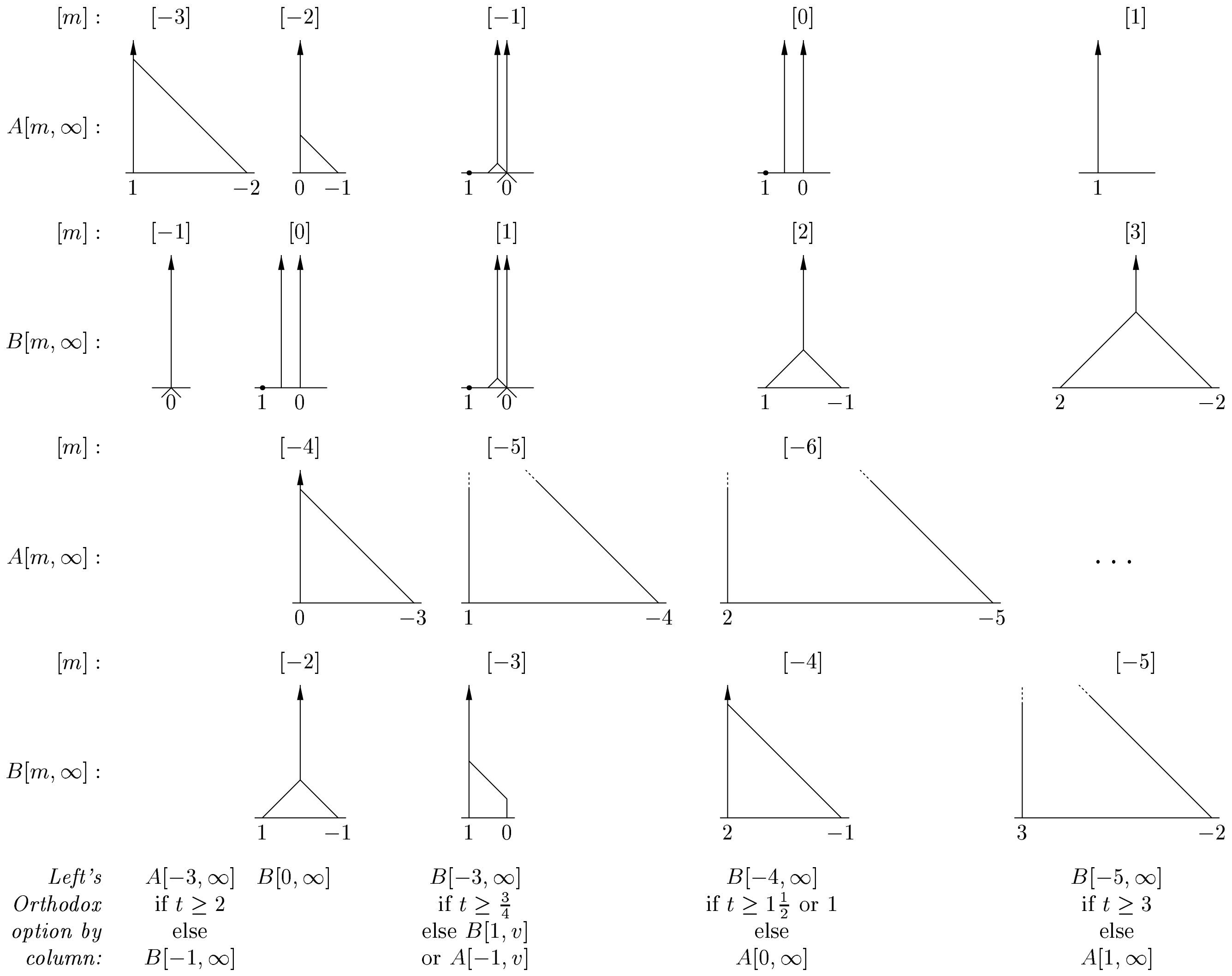


FIGURE 9. An array of thermographs. All horizontal lines (as seen with the page sideways) correspond to $t = 0$. The diagrams for $A[-1, \infty]$, $A[0, \infty]$, $B[0, \infty]$, and $B[1, \infty]$ are actually superpositions of two different thermographs, corresponding to even or odd n.

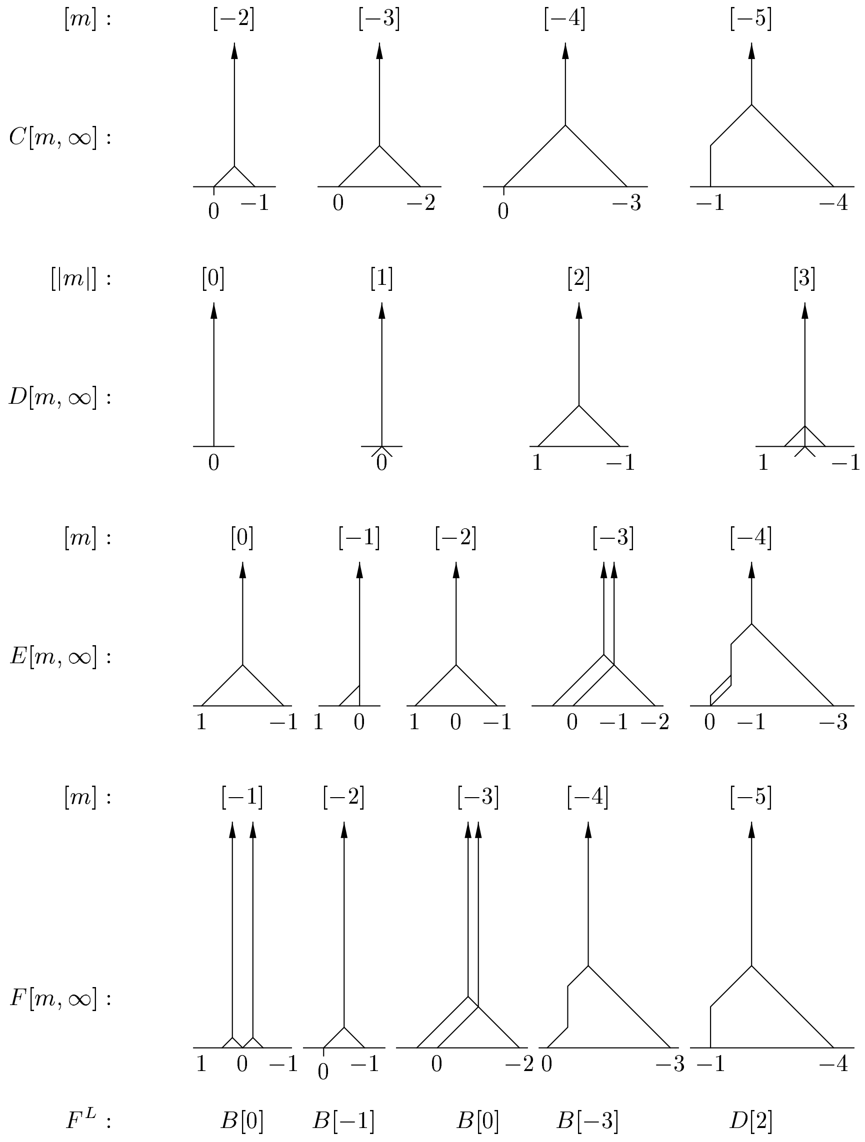
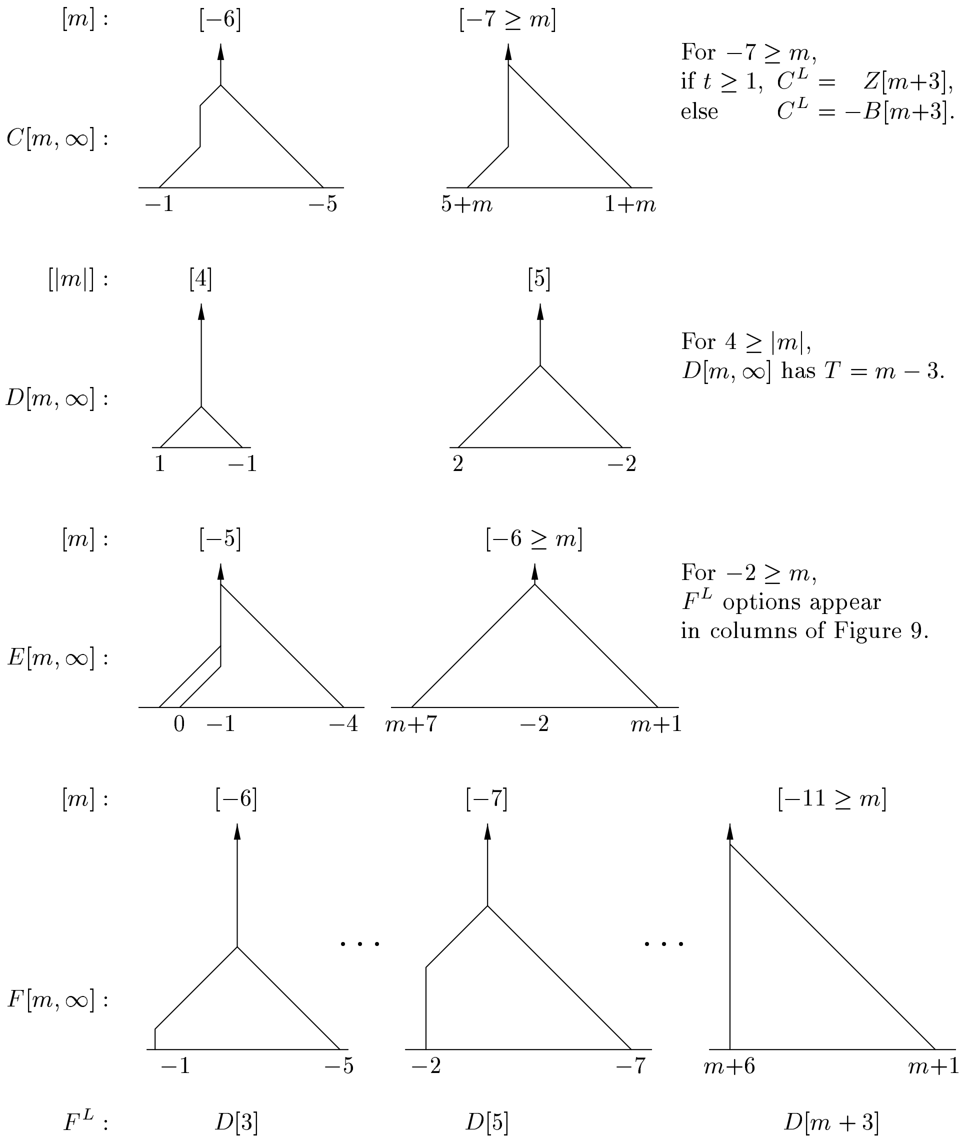


FIGURE 10. More thermographs. As in Figure 9, some of the diagrams are superpositions of two thermographs, for even and odd n. Such doublings are immediately apparent except in the case of $E[-1, \infty]$, where one thermograph is exactly as shown, whereas the other is just the mast starting at $t = 0$. When jaggedness is favorable, Left can improve on

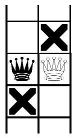


$F[-3, v] = \frac{1}{2}|{-2}$ by a toss-away. We call the resulting position $\mathcal{F}$. It looks exactly like $F[-3, v]$ in the vicinity of the Amazons, but it contains a block located one square away from the distant edge of the board. Its value is $\frac{3}{4}|{-2}$. This is the *only* case in which a toss-away yields an improvement in the thermograph.

positions of type A or B that might result from an opening Black move against a centrally situated White Amazon. As summarized by the equations at the bottom of each column of Figure 9, Black's orthodox choice within each column depends on the temperature. In each of the first four columns, the result can also depend on the parity of n. In the rightmost column and beyond, the White Amazon is far enough from the center for Left to reach $A[m,\cdot]$, where m is a positive integer. So these columns all lie in the region shown in Figure 5 on page 13, where on a sufficiently large board, Black's orthodox opening move is to $B[-m-4,\infty]$.

Except for the infinitesimals $A[-1,d]$, $B[-1,\infty]$, and $B[1,d]$, all other thermographs in Figure 5 have walls with slopes that are continuous at the temperature $t=0$. This is also true for the thermographs of almost all of the entries in Table 1, the exceptions being the infinitesimals and the value $E[2,9]$.

Thermographs of positions C, D, E, and F are presented in Figure 10. These results depend not only on A and B, but also on a particularly easy left follower of C, namely

$Z =$  .

It is easily seen that if k is the magnitude of m, then $Z[m,n] = k-1|k+1$, independent of n. We recommend that the diligent reader seeking to verify the thermographs in Figures 9 and 10 should work through them in alphabetical order.

Parity. The *parity* of an Amazons board position is defined as the parity of the number of empty squares eventually accessible to at least one Amazon.

Some game values can also be said to have a definite parity.

A game has *even* parity if it is zero or if all of its canonical followers are odd. It has *odd* parity if all of its canonical followers have even parity.

If a game has both odd and even followers, or mixed followers, then its parity is said to be *mixed*. Others are said to be *fixed*.

- The games 0, 2, 4, 6, ... are even.
- The games 1, 3, 5, 7, ... are odd. So are $*$, $2|-2$, and many others.
- The games $1|*$, $1|-1$, and many others are even.
- The games $\frac{1}{2}$, $\frac{1}{4}$, and $*2$ are mixed. More generally, all numbers except the integers have mixed parity, as do all nimbers except 0 and $*$.

In fact, game values of fixed parity are very restricted.

Table 1 reveals a strong tendency for odd boards to have values of odd parity, and for even boards to have values of even parity. The explanation for this tendency is straightforward. Most moves change the parity of the board. In fact, the only moves which preserve the parity of the board are those which seal off some empty region containing an odd number of squares. Hence, a mixed value such as $\frac{1}{2}$ or $\frac{1}{4}$ can occur only if it has a canonical position from which one player has a worthwhile move which seals off an odd number of squares. This observation helps explain why there are no worthwhile invading moves from $A[0, v]$ or from $B[0, d]$. In both cases, the seal would isolate an even number of squares in the top portion of the board, and that would not preserve the board's parity. The board's parity would become odd, suggesting odd values such as $*$ rather than even values such as 0.

Parity also plays an important role in at least four other games: Checkers, Blockbusting, Domineering and Go.

7. Orthodoxy and Canonical Theory

So far in this paper, we have emphasized orthodoxy and avoided any direct use of the fundamental results of combinatorial game theory. This theory, which resembles algebra much more than analysis, was pioneered by John Conway, then continued and expanded by many others (see [Nowakowski 1996], for example). Good expositions of the fundamentals can be found in [ONAG] and in [WW]. This theory emphasizes getting the last move in *any* environment. The minimal set of options which might be needed do so are called *canonical* moves. Perhaps the major differences between orthodoxy and the canons are philosophical. Canonical theory is tempted to work upward from 0 in the traditional way: 1, 2, 3, … . Orthodoxy is more tempted to begin at ∞ and work back down. Orthodoxy is more like classical minimax theory [von Neumann and Morgenstern 1944].

In general, orthodoxy is helpful for studying very hot games such as Amazons and Go. Professional Go players teach and practice their own rather imprecise version of orthodoxy, which is often sufficient to help them find good moves quickly. Efforts to teach them the canons have thus far met with only limited success. Canonical theory, on the other hand, is universal. It proves especially powerful at handling infinitesimals, including nimbers, about which orthodoxy has nothing useful to say.

For numbers, which have subzero temperatures, orthodox moves and canonical moves coincide. But at temperatures of 0 and higher, many games have unorthodox canonical moves, sometimes an embarrassingly large number of them.

One such example is Black's unorthodox opening option from S to the position G defined on page 12. This move is certainly not orthodox; it differs from option A by a term which decreases the mean by one. From an orthodox perspective, this move seems just as likely to cost Black three points as to gain her one. Yet, in a subsequent section, we shall show that Black's move from S to G is among her canonical choices.

All orthodox moves are canonical, but many canonical moves are unorthodox. Any canonical move on a game G will provide the only route to victory on $G + X$, for some suitably contrived value of the game X. A top-down perspective reveals that the most challenging case is inevitably the choice $X = -G$, the study of which leads to the canonical form. This can be rephrased to sound like a theorem in psychiatry: If you can deal with your own negative image, you can deal with anything else in the world!

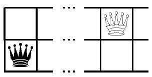
Only orthodox moves are needed to obtain the optimal payoff when G is played in a sufficiently rich environment. Even if the overall endgame has no cards, it may contain several independent board positions with which the local position G is summed. In such cases, it often happens that from the local perspective of G, moves in other summands serve the same role as moves in an environment consisting of cards. Orthodox moves are sufficient to win in many or even most reasonably rich environments. Unorthodox moves become necessary only in certain rather impoverished environments. The orthodox point of view also provides the basis for an accurate accounting system introduced by Berlekamp [1996]. Throughout the game, this accounting system maintains a record of the current "forecast", which consists of the mean, adjusted by half the ambient temperature to account for whoever currently has "sente". As detailed in the reference, this accounting system allows one to assess the costs and potential payoffs of every unorthodox move. Many a contemplated move of this type can be discarded from consideration because its immediate cost is seen to exceed its maximum potential benefit.

Programs which must find a winning play in an impoverished sum of several endgame positions need to be prepared to search the full canonical trees. However, it is well-known [Knuth and Moore 1975] that the fastest search algorithms are those which pursue the more promising moves first. Although to date this result has been used primarily with game trees generated by artificial intelligence heuristics, we are confident that it will also apply to Go and Amazon endgames. Complete searches of canonical trees should go much faster if orthodox moves are considered first.

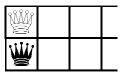
Creeping canonical complexity. One measure of the complexity of a game is its *birthday*, recursively defined as one more than the birthday of its youngest follower. There are several different kinds of birthdays, depending on the set of allowed followers and on the initial conditions that specify which games have the birthday of zero. Reasonable choices of the initial condition include the empty set, the set of integers, and the set of numbers. Reasonable choices of allowed followers include all (formal) followers, canonical followers, and orthodox followers.

One test to determine whether a follower is canonical is to play the sum of the game, G, with its negative image, $-G = -G^R|-G^L$. The goal is to get the last move. If White can successfully respond to Left's opening move from $G-G$ to G^L-G by continuing on to $G^{LR}-G$, then Left's move from G to G^L is said to be *reversible*. Or, if White can successfully respond to Left's opening move from $G-G$ to G^L-G by playing a significantly different option on $-G$, then Left's move from G to G^L is said to be *dominated*. Canonical moves are those to which the opponent can successfully respond *only* by playing to the precise negative image or a position strictly equivalent to it.

We now show that some games which are very simple from an orthodox perspective can be quite complex, canonically. To this end, we now claim that both of the following Amazons' positions have very large numerical canonical birthdays:

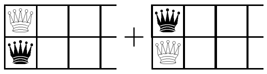
$V =$ 

and

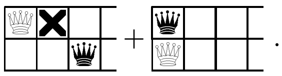
$W =$  .

A position of type V is the negative of an essential formal follower of $A[0, d]$ and $B[0, v]$ in Table 1. It is the initial move of a Right invasion which is the first position in a reversible sequence leading to 0. The position $-V$ is a crucial ingredient of the $\frac{1}{2}$ point values of $A[0, d]$ and $B[0, v]$.

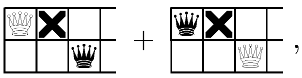
Let us set up the position $W - W$:



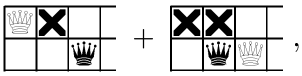
Left can play to



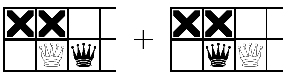
A thorough study of this position reveals that Right's *only* viable response is the negative image move to



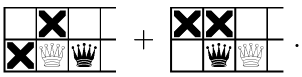
from which Left can continue to



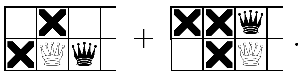
and again, Right's only winning response is to



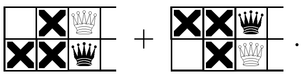
or the equivalent



Assuming the latter choice, Left can continue to

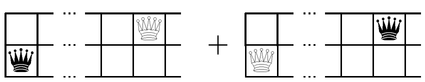


And again, a close investigation reveals that Right's only choice is the negative image response to

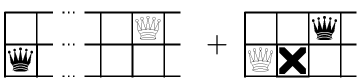


So we conclude that all three of Left's moves are canonical. Clearly, Left can force this loop to be repeated again and again until the canonical value becomes a number only when the Amazons get close to the far side of the board. From this we conclude that, if the initial position W has N empty squares, then its numerical birthday is at least $\frac{3}{4}N - c$, where c is a constant independent of N.

Next, we then consider the game $V - V$:

 ,

from which Left can open to

 .

Like the first follower of $W - W$ above, this is a position from which White's only winning choice is the negative image, which we have already considered above. Evidently, the canonical form of V, like W, must have numerical birthday which for a large number of empty squares, N, grows at least as fast as $\frac{3}{4}N$.

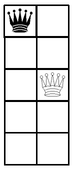
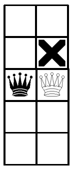
Canons Embraced! Despite the potential complexities exhibited in the previous subsection, canonical theory is a rich and powerful methodology. When properly used, orthodoxy and the canons are deeply intertwined.

We now discuss some of the subtler results presented in Table 1. This discussion requires canonical thinking, with which we henceforth assume the reader is familiar.

Invasions which beat $*$ to gain more than $\uparrow$. We now consider those columns of Figure 1 in which $|m| = 1$. Since the parity of this board is odd, and we are in a region in which values are simple and small, we expect the default value to be the small odd value, namely $*$. If there are no followers of mixed value, and if Left's best invading move is to V, then we might expect the value of the game to be

$$\{0, V|0\} = \begin{cases} * & \text{if all } V \ngeq *, \\ \uparrow * & \text{if some } V = * \text{ but no } V > *, \\ > \uparrow * & \text{if some } V > *. \end{cases}$$

Not surprisingly, we find almost no worthwhile invasions from $A[-1, v]$, or from $B[-1, d]$ or from $B[1, d]$. In all of these cases, any plausible invading move seals off an even number of squares on the top of the board. That ensures that the parity of the board position V is still even, so its value is very likely to be 0 rather than anything $\geq *$. But one case slips through this relatively strong parity barrier. From either $-B[-1, 7]$ or $-B[1, 9]$, Right can play an invading move which might yield this sequence:

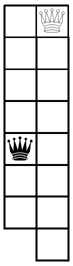
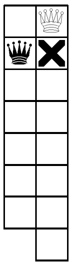
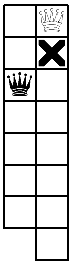
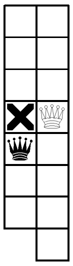
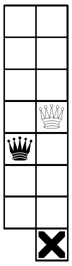
$V =$  , then $V^L =$  $= 0.$

This V^L is a truly exceptional game. Its board parity is odd; yet its value is 0. This same position underlies many of the exceptional values found in Figure 1 with $n = 8$, 9, and 10. Many of the exceptional values with smaller n are positions of this game.

If $|m| = 1$, all unexceptional worthwhile invasions from A or B lead to a board with a jagged bottom edge. The value of a typical Left invasion is

$$V = \{bigs|0\} > * = \{0|0\},$$

where "*bigs*" can assume a variety of values, depending on n. For example, from $B[1, 14]$ or $B[-1, 12]$ Left might invade as follows:

$V =$  $=$  ,  |  ,  , ... ?

$$= \quad 10 \quad , \{11|7, \dots\} \mid \quad 0 \quad , \ u * |{-1} < 10, \{11|8*\} \,\|\, 0.$$

Although we have shown on page 28 that the canonical value of this (not atypical) invasion is creepingly complex, we can give it an upper bound of $11*|0$, a typical case of a game we more christen with the generic term, $bigs|0$. This is sufficient to find an upper bound on its parent,

$$0, \{bigs|0\}\|0.$$

This value clearly exceeds

$$0, *\|0 = \uparrow *.$$

However, the excess is very small, as can be seen by selecting a large positive integer, N, and then playing

$$N \cdot \{0, \{bigs|0\}\|0\} + N \cdot (\downarrow *) + \downarrow.$$

This sum is soon seen to be negative. Left's dominant opening move is to replace the $(N+1) \cdot (\downarrow *)$ by $N \cdot (\downarrow *)$, to which Right responds by removing one of the $\{0, \{bigs|0\}\|0\}$. So the dominant line of play after N moves by each player leads to a value of $\downarrow$, and a win for Right. Thus, $\{0, \{bigs|0\}\|0\} + \downarrow *$ is a positive infinitesimal of higher order than up. So I have denoted the value of any such game as $u*$, with the understanding that u is a positive infinitesimal only very slightly larger than $\uparrow$. If sums and differences of such games are played with each other, the outcome may depend on the specifics of the *bigs*. However, all such games and their sums and differences have the same order relationships with all sums and differences of the other games in Table 1. To first order, u behaves exactly the same as $\uparrow$.

Hence, the dominant infinitesimal issue in compiling Table 1 is to distinguish the cases which have value $u*$ from other cases which have only the default value, $*$. This depends on whether or not Left can respond to Right's invasion to reach a position of value $\geq *$.

Picking the proper square on which to land the invader can involve even more subtle considerations. Including the squares occupied by the two Amazons, the total number of squares on the board remaining after an invasion from $-B[1,n]$ or $-B[-1,\, n{-}2]$ is $n+1$. If n is 3 mod 4, the horizontal centerline of the new board runs between a pair of rows; in the other three cases, this centerline runs through a row. When n is even, the boardsize is odd. Since the top edge of the board is smooth, the bottom edge of the board must be jagged. In almost all such cases, Right should not undertake an invasion which places herself on the row which straddles the centerline, because Left can then respond to a position of type A or B and value $+\frac{1}{2}$. Right's correct invasion typically places herself on the row directly below the centerline. The only exception is when $n = 10$ and the jag at the bottom of the board favors Left. In this case, if Right invades at the point directly below the row containing the centerline, Left can respond to $E[2,5] = 1|0 > *$. However, in this case Right can successfully invade to the row containing the centerline, because Left's defensive response to $B[0,4]$ yields the exceptional value of 0 which is $\not\geq *$.

Moves which gain $\{\frac{1}{2}|0\}*$. Finally, the columns of Table 1 with $|m| = 1$ show several classes of positions in which Left has a non-invading option that yields a value of $\frac{1}{2}|0$ rather than $*$. From $A[-1,v]$, Left can descend one square and then throw-away upwards to reach $B[0,v] = \frac{1}{2}$. Similarly, from $B[1,v]$ Left can ascend one square and then throw-away downwards, yielding $A[0,d]$, whose value is typically $\frac{1}{2}$. Finally, from $E[-1,v]$ Left can reach $B[0,v]$ by descending one square and shooting northeastward.

Of course, a move to $\frac{1}{2}$ dominates any move to an infinitesimal such as $u*$.

Amazonian incentives. From a game $\{G = G^L|G^R\}$, Left's *incentives* are all games of the form $G^L - G$, and Right's incentives are all games of the form $G - G^R$. Any dominant move on a sum of games must have a dominant incentive. Fortunately, incentives can often be ordered. Table 2 shows a top-down ordering of incentives of many of the values which appear in Table 1.

A game of the form $x|y$, where x and y are numbers, is called a *switch*. All of the incentives in the above list are *switch-ish*.

TABLE 2. Ordered incentives.

Incentive:	$2\|0$	$\frac{5}{4}\|0$	$1u*\|0$	$1*\|0$	$\frac{1}{2}\|0$	$u*$	$\uparrow*$	$*$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1
Example:											
Type:	D	E	E	E	A	B	C	B	B	A	A
m:	2	2	2	2	-1	-1	-1	-1	-1	0	1
n:	$\cdot$	3	d	v	v	v	1	d	1	d	$\cdot$

This list is totally ordered; either player will always prefer a move whose incentive appears higher on this list to another which appears lower. However, some incentives which we have not listed are incomparable with some of those on this list. The incentive of $E[2,5]$ is $1|0$, which is incomparable with both $1u*|0$ and $1*|0$. $C[-1,1]$ has incentive $\downarrow$, which is incomparable with $*$. No local rule alone can always correctly resolve a choice between two incomparable incentives. There are some environments where either wins and the other loses.

The only entry in Figure 1 which is not Switch-ish is $B[2,\cdot]$. From this position, Right's dominant move is to plod upward to a value of -1. Left may also plod upward to a value of $+1$. This is the only move which achieves the correct leftwall at temperature 0. However, this move does not dominate Left's invading jump downward to a value such as $bigs|0$ or $bigs|*$. Nor do Left's invading moves reverse out. Although they have no effect on the scores of this game alone, these moves do indeed affect the scores of sums such as $B[2,\cdot]+E[0,\cdot]$. In particular, if some $big > 1$, then Left can achieve a score of 1 by playing first from the sum,

$$1, bigs|0 \,\|\, -1 \;+\; 1|-1$$

even though

$$1|-1 \;+\; 1|-1 = 0.$$

APPENDIX A: PROOF THAT FRAMES = WALLS

Preliminaries. If x, a_1, a_2, … are real numbers, we say that

$$x \in \text{range}\{a_1, a_2, \ldots, a_n\}$$

if $\min\{a_i\} \le x \le max\,\{a_i\}$. We will make no effort to order the a_i; in the cases of interest all of them will be fairly small.

It is not hard to show that if $x = \min\{a,b\}$ and $y = \max\{c,d\}$, then $x+y \in \text{range}\{a+c, b+d\}$. One easy proof is simply to check all four possible cases.

The proof. Let G be an arbitrary board game, and let N be the length of a maximal sequence of plays which can be made in G. Let S be a stack of cards of temperature t and decrement δ, terminating with at least N cards of value -1 before the terminal komi of $-\frac{1}{2}$. Let S' be the same stack with the top card removed, S'', with the top two cards removed, etc. The temperature of S' is $t' = t - \delta$.

We now assert that

$$\begin{aligned}\mathrm{RF}(G,t) - \mathrm{RP}(G+S) \in \mathrm{range}\big\{&\mathrm{LF}(G^R,t) + t + \epsilon_1 - \mathrm{LP}(G^R+S),\\ &\mathrm{LF}(G,t') + t - \mathrm{LP}(G+S')\big\},\end{aligned}$$

where $-\delta \le \epsilon_1 \le \delta$. (Successive ϵ_i's that appear below also satisfy this condition.) To prove this assertion, we note that

$$\begin{aligned}\mathrm{RF}(G,t) &= \begin{cases}\mathrm{LF}(G^R,t)+t & \text{if } t \le \tau,\\ \mathrm{LF}(G^R,\tau)+\tau & \text{if } t' \le \tau \le t,\\ \mathrm{LF}(G,t') & \text{if } \tau \le t'\end{cases}\\ &= \min\big\{\mathrm{LF}(G^R,t) + t + \epsilon_1,\, \mathrm{LF}(G,t')\big\}\end{aligned}$$

and

$$-\mathrm{RP}(G+S) = \max\big\{-\mathrm{LP}(G^R,t),\, t - \mathrm{LP}(G+S')\big\}.$$

By interchanging the roles of Right and Left, and replacing G by G^R, we obtain

$$\begin{aligned}\mathrm{LF}(G^R,t) - \mathrm{LP}(G^R+S) \in \mathrm{range}\big\{&\mathrm{RF}(G^{RL},t) - t + \epsilon_2 - \mathrm{RP}(G^{RL}+S),\\ &\mathrm{RF}(G^R,t') - t - \mathrm{RP}(G^R+S')\big\}.\end{aligned}$$

Again interchanging the roles of Right and Left, but now replacing t by t' and S by S', we obtain

$$\begin{aligned}\mathrm{LF}(G,t') - \mathrm{LP}(G+S') \in \mathrm{range}\big\{&\mathrm{RF}(G^L,t') - t' + \epsilon_3 - \mathrm{RP}(G^L+S'),\\ &\mathrm{RF}(G,t'') - t' - \mathrm{RP}(G+S'')\big\}.\end{aligned}$$

Combining the prior ranges yields

$$\begin{aligned}\mathrm{RF}(G,t) - \mathrm{RP}(G+S) \in \mathrm{range}\big\{&\mathrm{RF}(G^{RL},t) + t - t + 2\epsilon_4 - \mathrm{RP}(G^{RL}+S),\\ &\mathrm{RF}(G^R,t') + t - t + \epsilon_5 - \mathrm{RP}(G^R+S'),\\ &\mathrm{RF}(G^L,t') + t - t' + \epsilon_6 - \mathrm{RP}(G^L+S'),\\ &\mathrm{RF}(G,t'') + t - t' - \mathrm{RP}(G+S'')\big\}.\end{aligned}$$

We next note that $\mathrm{RP}(S'') = \mathrm{RP}(S) - \delta$, and that $\mathrm{RP}(S'') \le \mathrm{RP}(S') \le \mathrm{RP}(S)$. Making the appropriate adjustments to the prior expression, we obtain

$$\begin{aligned}\mathrm{RF}(G,t) - \mathrm{RP}(G+S) + \mathrm{RP}(S) \in \mathrm{range}\big\{&\\ &\mathrm{RF}(G^{RL},t) + 3\epsilon_7 - \mathrm{RP}(G^{RL}+S) + \mathrm{RP}(S),\\ &\mathrm{RF}(G^R,t') + 3\epsilon_8 - \mathrm{RP}(G^R+S') + \mathrm{RP}(S'),\\ &\mathrm{RF}(G^L,t') + 3\epsilon_9 - \mathrm{RP}(G^L+S') + \mathrm{RP}(S'),\\ &\mathrm{RF}(G,t'') - \mathrm{RP}(G+S'') + \mathrm{RP}(S'')\big\}.\end{aligned}$$

This expression now forms the basis for an inductive proof of the following assertion:

$$\left|\mathrm{RF}(G,t) - \mathrm{RP}(G+S) + \mathrm{RP}(S)\right| \le 3N\delta.$$

For if not, let G be a counterexample with a minimal value of N, so that the assertion is true for all t for G^R and G^L, which have length at most $N-1$. That yields bounds no more then $3N\delta$ for each of the first three elements of the set that constrains the range. Finally, let t be the minimal temperature for which the assertion fails. Since $t'' < t$, the fourth element in the range has magnitude no greater than $3N\delta$, and so the induction is completed.

Similarly, it also follows that

$$\left|\mathrm{LF}(G,t) - \mathrm{LP}(G+S) + \mathrm{LP}(S)\right| \le 3N\delta.$$

Evidently, as δ approaches 0, $\mathrm{LP}(G+S) - \mathrm{LP}(S)$ approaches $\mathrm{LF}(G,t)$. Since the left wall was defined as this limit, we have shown that

$$\mathrm{LW}(G,t) = \mathrm{LF}(G,t) \quad \text{and} \quad \mathrm{RW}(G,t) = \mathrm{RF}(G,t). \qquad \text{Q.E.D.}$$

Appendix B: Problem Solution

Figure 11 shows the position of the board after the eleventh move of the main line of the solution to the problem presented in Figure 1. The burned squares are numbered in order; White's shots are odd and Black's are even. White's startling first move illustrates the toss-away to $-\mathcal{F} = 2|{-\frac{3}{4}}$, slightly better than the throwaway to $-F[3,v] = 2|{-\frac{1}{2}}$. White 1 contacts Black and then finds a surprisingly clever square to burn. Black 2 contacts White diagonally. White 3 exemplifies the infamous "Option B". Black 4 moves onto the same row as White. From this information, the reader can infer where each Amazon went on each of her moves. After move 11, the canonical values of the four regions add up to $* - \frac{1}{2} - 1 + 1$, a win for White. Note the lack of alternating play within some local regions. In one region, White makes the first four consecutive moves (1, 5, 9, and 13), sealing off odd

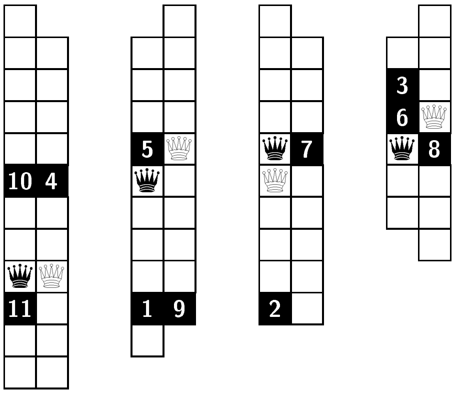


FIGURE 11. Solution to the problem of Figure 1.

portions at both the bottom and the top of the board, while Black must repeatedly play elsewhere.

Even though there are no cards, all of the moves in this main line of the solution are orthodox, and consistent with sentestrat. As often seems to happen, these four games provide "rich enough" environments for each other.

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ONAG: an abbreviation for [Conway 1976].

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WW: an abbreviation for [Berlekamp, Conway, and Guy 1982].

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