Hot Games and Theory

**CRAM** is also called **Impartial Domineering**. Each player, in his turn, places a domino in either orientation onto any available pair of adjacent empty squares. Last move wins.

Problems:

For each of the following sequences of Cram positions,

1. Express $G_n$ recursively, using nimsum(s) $\oplus$ and mex.

2. For $n = 0$ to 10, determine the nimbers $G_n$.
   [Hint: Recur $B$ and $C$ together if you wish.]

Solutions.

1. For (A), $A_n = \text{mex}_k A_k \oplus A_{n-2-k}$. For (B) and (C),

\[
B_n = \begin{cases} 
\text{mex}_k \{B_k \oplus C_{n-2-k}, B_k \oplus C_{n-1-k}\}, & \text{if } n \text{ is even} \\
\text{mex}_k \{B_k \oplus B_{n-2-k}, B_k \oplus B_{n-1-k}\}, & \text{if } n \text{ is odd}
\end{cases}
\]

\[
C_n = \begin{cases} 
\text{mex}_k \{C_k \oplus B_{n-2-k}, C_k \oplus B_{n-1-k}\}, & \text{if } n \text{ is even} \\
\text{mex}_k \{C_k \oplus C_{n-2-k}, C_k \oplus C_{n-1-k}\}, & \text{if } n \text{ is odd}
\end{cases}
\]

2. Recursively,

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_n$</th>
<th>excludants</th>
<th>$n$</th>
<th>$B_n$</th>
<th>$C_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1,0</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1,1</td>
<td>5</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>2,1,0</td>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0,2,0</td>
<td>7</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>3,0,3,0</td>
<td>8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>1,3,1,3</td>
<td>9</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>1,1,2,1,0</td>
<td>10</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
For $n \geq 2$, $B_n = B_{n-2}$ and $C_n = C_{n-2}$.

\[ B_n \equiv \left\lfloor \frac{n}{2} \right\rfloor \mod 2; \quad C_n \equiv \left\lfloor \frac{n}{2} \right\rfloor \mod 2.\]
1  $N \times 1$ Konane Problems

In Konane, Left plays X’s and Right plays O’s. A legal move must jump over an adjacent opponent’s piece in a horizontal or vertical direction and land on an empty square. The jumpee is removed from the board. Multiple straight-line jumps are allowed, but not compulsory. The player unable to move loses.

For each of the following infinite sequences of single-row Konane positions, find the canonical forms: (A recursive answer MAY be acceptable if appropriate).

$$A_n = \begin{array}{cccccccc}
\text{O} & \text{X} & \text{O} & \text{X} & \ldots & \text{O} & \text{X} \\
\text{2n contiguous occupied squares} \\
\end{array}$$

$2n+2$ squares on the board

$$B_n = \begin{array}{cccccccc}
\text{O} & \text{X} & \text{O} & \text{X} & \ldots & \text{O} & \text{X} & \text{O} \\
\text{2n+1 contiguous occupied squares} \\
\end{array}$$

$2n+3$ squares on the board

$$C_n = \begin{array}{ccccccc}
\text{X} & \text{O} & \text{O} & \ldots & \text{O} & \text{O} \\
\text{repeated n times} \\
\end{array}$$

$2n+2$ squares on the board

$$D_n = \begin{array}{cccccccc}
\text{O} & \text{X} & \text{O} & \text{X} & \ldots & \text{O} & \text{X} & \text{X} & \text{O} \\
\text{2n contiguous occupied squares} \\
\end{array}$$

$2n+5$ squares on the board

Solutions.

$$A_0 = 0, \quad A_n = A_{n-1} | A_{n-1} = \begin{cases} *	ext{, if } n \text{ odd} \\ 0\text{, if } n \text{ even} \end{cases}$$

$$B_n = \{n - 1\} = n, \ (n \geq 0).$$


<table>
<thead>
<tr>
<th>$n$</th>
<th>$D_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>*</td>
</tr>
<tr>
<td>2</td>
<td>*</td>
</tr>
<tr>
<td>odd ≥ 3</td>
<td>1</td>
</tr>
<tr>
<td>even ≥ 4</td>
<td>1*</td>
</tr>
</tbody>
</table>

$$C_0 = 0$$

$$C_1 = *$$

$$C_n = \{D_k\}_{k<n}|0$$

$$\Rightarrow C_2 = 0, * \mid 0 = \uparrow *$$

$$C_3 = 0, *, \uparrow * \mid 0 = 0, \uparrow, * \mid 0$$

$$\ldots$$

Hence, $C_n = \{0, C_{n-1} \mid 0\}$ is canonical for $n \geq 1$. since $D_1 < D_2 < \cdots < D_{n-1} < D_n$ yet all are confused with 0.
2 Amazon Problems

\[
\begin{align*}
A &= \begin{array}{|c|c|c|c|c|}
\hline
& & & & \\
\hline
& & & R & L \\
& & & & \\
\hline
\end{array} \\
B &= \begin{array}{|c|c|c|c|c|}
\hline
& & & & \\
\hline
& & R & L & \\
& & & & \\
\hline
\end{array} \\
C &= \begin{array}{|c|c|c|c|c|}
\hline
& & & & \\
\hline
& & R & L & \\
& & & & \\
\hline
\end{array}
\end{align*}
\]

Each of the above 3 Amazon games may be represented as a game tree of the following form (assume \( J \) is at least as hot as \( K \)):

```
G
   H
   
J
   K
```

For \( G = \) game \( A \) above:

4A) Evaluate the leaves of the tree, and express the game in canonical form.

5A) On the same \( k = 0 \) axis, plot the thermographs of \( G, H \) and \( K \). Show the mean \( \mu \) and freezing temperature \( \tau \) of each.

6A) What is the infinitesimal \( G_\tau - \mu \)?

4B-6B) Repeat 4-6 with \( G = B \).

4C-6C) Repeat 4-6 with \( G = C \).

**very hard:**

7) Consider the sum \( D = A + C - 2B \). On this sum, would you choose to play first or second or Left or Right?
8) What are Leftscore($D$) and Rightscore($D$)?

Solutions.

4) 

\[ A = \begin{array}{c|c|c|c}
\text{Leftscore} & \text{Rightscore} & \text{Game} & \text{Result} \\
\hline
6 & 4 & -4 & (=) 4 & -4 \\
\hline
7 & 3 & -3 & (=) 5 & -5 \\
\hline
4 & 2 & -2 & (=) 5 & -5 \\
\hline
\end{array} \]

B = \begin{array}{c|c|c|c}
\text{Leftscore} & \text{Rightscore} & \text{Game} & \text{Result} \\
\hline
5 & 3 & -3 & (=) 5 & -5 \\
\hline
6 & 4 & -4 & (=) 5 & -5 \\
\hline
3 & 2 & -2 & (=) 5 & -5 \\
\hline
\end{array} \]

C = \begin{array}{c|c|c|c}
\text{Leftscore} & \text{Rightscore} & \text{Game} & \text{Result} \\
\hline
4 & 2 & -2 & (=) 5 & -5 \\
\hline
5 & 3 & -3 & (=) 5 & -5 \\
\hline
2 & 1 & -1 & (=) 5 & -5 \\
\hline
\end{array} \]

5) 

6) 

\[ A_3 - 7 = 0||0||0|-2 = 0 + 2 \]
\[ B_3 - 6 = 0||0||0|0=0 \uparrow \uparrow \downarrow \* \]
\[ C_{24} - \frac{1}{4} = \* \]

7) Choose to play FIRST.
Temperatures of positions of $D$ are 4, 3, $2\frac{3}{7}$, $2\frac{1}{2}$, 2. Since the big gap is between $t = 2$ and $t = -1$, Right should play to get last move at $t \geq 2$. To do this, he should be prepared to concede means on positions of $C$, and merely use $C - B \leq 1$.

Since $\uparrow \,* > 0|+,$ Right can get the last big move on EACH of the pairs $A - B$ and $C - B$, while conceding $+\frac{3}{4}$ adjustments on mean $C$.

8) Leftscore($D$) = 1; Rightscore($D$) = -1.

Full credit simply for observing that $\mu(D) = \frac{1}{4} > 0$, so Left playing first can win.
(1) The following 3 Amazon positions are on a $2 \times 14$ board. Plot the thermograph of each, showing coordinates of all points where either wall of the thermograph changes slope.

![Thermograph of A](image1)

![Thermograph of B](image2)

![Thermograph of C](image3)

(2A) Express $\int_1^1 \frac{3}{8}$ in terms of slashes, integers, and stars.

(2B) Plot the thermographs of $\int_1^1 \frac{3}{8}$, and of its Left and Right followers.

(2C-E) Express each of the following in terms of slashes, integers and stars.

(2C) $\int_{1\frac{3}{8}}^1$

(2D) $\int_1^{1\frac{3}{8}}$

(2E) $\int_1^{1\frac{3}{8}}$

(3) Define an environment, $E = \pm 5 \pm 3 \pm 1$.

Define games $A = f^6 * = \pm 6$ and $B = f^6 \uparrow = 6 \parallel 0 \mid 12$.

(3E) Compute the thermograph of $E$.

(3A) Compute the thermograph of $A + E$.

(3B) Compute the thermograph of $B + E$. 

7
(3C) On each thermograph, where $G$ is the game, circle $\text{Leftscore}(G_t)$ and $\text{Rightscore}(G_t)$ at the points where $t = 0, 2, 4, 6$.

(4A) Find the canonical form of this single-jump Konane position (assuming multiple jumps are illegal).

\[ J = \begin{array}{cccc}
\times & \text{O} & & \\
\text{O} & & & \\
& \times & & \\
& \text{O} & & \\
\end{array} \]

(4B) Compare $J$ with integer multiples of $\uparrow$.

---

**Grading Plan**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>( \sum )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>30</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>10</td>
<td>21</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>10</td>
<td>20</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>151</td>
<td></td>
</tr>
</tbody>
</table>
Solutions.

(1A) \(\{19 - 5 \mid -15 + 9\} = \{14 \mid -6\} = 4 \pm 10; A_{10} = 4 \ast.\)

(1B) \(\{18 - 6 \mid 15 - 8 \mid 8 - 15\} = \{12 \mid 7 \mid -7\}; B_5 = \{7 \mid 7 \mid 3\} = 7 + [4].\)

(1C) \(\{22 - 3 \mid A, B\} = \{19 \mid A, B\}; C_6 = 13 \ast.\)

(2A)

\[
\int_1^1 \frac{1}{2} = \int_1^1 \{0 \mid 1\} = 1 \mid 0.
\]

\[
\int_1^1 \frac{1}{4} = \int_1^1 \left\{0 \mid \frac{1}{2}\right\} = \left\{1 + \int_0^0 \left\lfloor -1 + \int_0^0 \frac{1}{2}\right\rfloor\right\} = \{1 \mid 0 \mid -1\}.
\]

\[
\int_1^1 \frac{3}{8} = \int_1^1 \left\{\frac{3}{4} \mid \frac{1}{4}\right\} = \left\{1 + \int_0^1 \left\lfloor -1 + \int_0^0 \frac{1}{2}\right\rfloor\right\} = \{2 \mid 1 \mid 0 \mid 0 \mid -1\}.
\]
(2C-2E)

\[ \int_{1/2}^{3/8} t = \int_{1/4}^{1/2} t + \int_{1/2}^{1} t + \int_{1}^{3/8} t + \int_{3/8}^{1} t \]

(2C) \[ \int_{1* \ 3/8}^{1*} = 2 \ || \ 1 \ * \ ||| \ 0 \ ||\ -1 \ * \ . \]

(2D) \[ \int_{1}^{1* \ 3/8} = 2 \ || \ 1 \ * \ ||| \ 0 \ ||\ -1 \ . \]

(2E) \[ \int_{1*}^{1* \ 3/8} = 2 \ || \ 1 \ * \ ||| \ 0 \ ||\ -1 \ * \ . \]

(3E), (3A) \[ \Delta \int^{6} \ * > \Delta \int^{5} \ * > \Delta \int^{4} \ * > \Delta \int^{3} \ * > \Delta \int^{2} \ * , \] so canonical play occurs in the order of temperature and we have these thermographs:
(3B) $\Delta^R \int^6 \updownarrow = 6 + \int^6 \updownarrow \ast > 6 + \int^6 \ast = \Delta \int^6 \ast$, so Right will play on B at his first opportunity. But since $\Delta^L \updownarrow = \downarrow \neq \ast = \Delta^L \ast$, Left might prefer to play a switch. The switch turns out to be Left’s better opening move at $2 < t < 4$, where the dotted line shown below in that region is dominated.

Since the environment is enriched, in all cases we have the same (regular) values at the circled points.

$$J = \begin{array}{c|c|c|c|c}
D & X & O & A \\
E & & & \\
F & & & \\
B & X & & \\
C & & & \\
\end{array}$$

$$J = \{ A, B, C \mid D, E, F \}$$
$$= \{ \uparrow, \ast, \ast \mid \ast, \ast, \uparrow \ast \}$$
$$= \{ \uparrow, \ast \mid \ast \} > 0$$
whence Left’s move to * reverses out and $J = \uparrow \mid \ast$.

$(4B) \uparrow> J > \uparrow.$
Blockbusting

Blockbusting is a partizan game played on an $n \times 1$ strip of squares by two players, called L and R (for Left and Right, or bLue and Red). The squares are called “parcels”. Each player, in turn, claims one previously unclaimed parcel and colors it with his color. The game ends when all parcels have been colored, and the score is the equal to the number of parcel boundaries which have been colored blue on both sides. In other words, Left seeks to maximize the number of neighbouring Left-Left pairs while Right seeks to minimize the same number. No points are awarded to either player for adjacent Right-Right pairs.

In one interpretation, Left and Right may be viewed as rival real estate agents buying up all of the real estate parcels on a new street. Left is a segregationist who seeks to place his clients next to neighbors of his same color; Right is an integrationist who seeks to break up bLue-bLue neighbors.

At a typical stage of play, the street is broken up into consecutive sequences of available parcels, and each end of each such sequence is colored according to the color of the adjacent (colored) parcel. Thus, there are three types of available stretches of length $n$: $LnL$, $LnR$ and $Rn$. (The fourth type, $RnL$ is easily seen to be equal to $LnR$.)

Following standard techniques, we may view these positions as games which satisfy the recursions shown below. The scoring rule is embedded in the initial conditions. The solution to this recursion, for $n \leq 2$, is also shown in the table.

Initial conditions: $L0L = 1; L0R = R0R = 0$

Recursion: For $n > 0$,  
\[
LnL = \{L(n-1-k)L + LkL \mid L(n-1-k)R + LkR\}, \quad 0 \leq k < n, \\
LnR = \{L(n-1-k)R + LkL \mid R(n-1-k)R + LkR\}, \quad 0 \leq k < n, \\
RnR = \{L(n-1-k)R + LkR \mid R(n-1-k)R + RkR\}, \quad 0 \leq k < n,
\]

<table>
<thead>
<tr>
<th>$n$</th>
<th>$LnL$</th>
<th>$LnR$</th>
<th>$RnR$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Problem:

(a) Continue this table to at least $n = 7$.
(b) Compute thermographs of all positions found.
(c) After a slightly irregular start, it soon settles into a repetitive pattern. Can you guess what it is?
(d) Can you find any sort of “parity” pattern here?
5 Thermograph and Confusion Interval

Let $G = \{51 + (\pm 50) \parallel \{0, \pm 50 \mid -20\}\}$.
(a) Draw the thermograph of $G$.
(b) For $n = 1, 2, 3, \ldots, 51$, determine the confusion interval of $n \cdot G$.

**Note**: Full credit will be given for determining the Left and Right stops $L(nG)$ and $R(nG)$; it is unnecessary to compare $n \times G$ with these endpoints of the confusion interval.
6  Essentials

Winning Moves: Each part of this problem is a game whose value is the sum of all values in the list shown. You are Right and it is your turn. Determine ALL winning moves, assuming each component is in canonical form.

A:  1/2
    5/8
    3/4
    -2

B:  \uparrow
    \downarrow
    \uparrow
    *

C:  3 \cdot \uparrow
    \downarrow

D:  -30^2
    -20^4
    0^2| + 2^{2_2}
    0^3| + 1
    0^2| + 3

E:  3|0
    1| - 1
    0| - 1
    +3
    -4

F:  *5
    *6
    *7
    *8

G:  0, * | 0, *, *2
    *5
    * = remote star

H:  0, *, *2, *3, *4 | 0, *, *3
    *

Solutions.
(1A) \frac{3}{4} \rightarrow \frac{3}{4}.
(1B) * \rightarrow 0, or
* \rightarrow * (dominant).
(1C) 3 \cdot \uparrow \rightarrow \uparrow *.
(1D) 0^3| + 1 \rightarrow 0^2| + 1.
(1E) 3|0 \rightarrow 0, or
+3 \rightarrow 0| - 3 (dominant).
(1F) *8 \rightarrow *4.
(1G) *5 \rightarrow anything,
* \rightarrow anything EXCEPT *4, *5.
(1H) 0, *, *2, *3, *4 | 0, *, *3 \rightarrow *
0, *, *2, *3, *4 | 0, *, *3 \rightarrow *3
## 7 Games Born on Day 2

The 22 games born on Day 2:

<table>
<thead>
<tr>
<th>Fuzzy</th>
<th>Positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negative</td>
<td>0</td>
</tr>
</tbody>
</table>

where the unit boxes comprise of both numbers and infinitesimals.
**Problem.** Draw a graph showing the partial ordering among all games born on or before Day 2. The names of these 22 games are as follows:

<table>
<thead>
<tr>
<th>$G^L$</th>
<th>$G^R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0, *</td>
</tr>
<tr>
<td>±1</td>
<td>0</td>
</tr>
<tr>
<td>0, *</td>
<td>±1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>*</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
</tr>
</tbody>
</table>

Solution.
Games born on day 2 grouped as clusters of games infinitesimally close to each other.
90 Games from Day 3

Problem. Consider the following table of 90 games. For each game which is number-ish, express the game as a sum of a number and one or more infinitesimals (e.g. $\frac{1}{2} \uparrow*$). For each game which is HOT, fill in the table with an equation stating the game’s temperature.

The sample row for $GL = \frac{1}{2}$ shows the desired format.

<table>
<thead>
<tr>
<th>$G^L$</th>
<th>$G^R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$-\frac{1}{2}$ 0 0</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{2}$ $t = \frac{1}{2}$ $t = \frac{3}{4}$ $t = \frac{1}{4}$ $t = \frac{3}{2}$</td>
</tr>
<tr>
<td>1*</td>
<td>$\frac{3}{2}$ $t = \frac{3}{2}$ $t = \frac{1}{2}$ $t = \frac{1}{4}$ $t = \frac{3}{4}$</td>
</tr>
<tr>
<td>↑</td>
<td>$t = \frac{1}{2}$ $t = \frac{1}{4}$ $t = \frac{1}{2}$ $t = \frac{3}{2}$</td>
</tr>
<tr>
<td>0</td>
<td>$t = \frac{1}{2}$ $t = \frac{1}{4}$ $t = \frac{1}{2}$ $t = \frac{3}{2}$</td>
</tr>
<tr>
<td>*</td>
<td>$t = \frac{1}{2}$ $t = \frac{1}{4}$ $t = \frac{1}{2}$ $t = \frac{3}{2}$</td>
</tr>
<tr>
<td>↓</td>
<td>$t = \frac{1}{2}$ $t = \frac{1}{4}$ $t = \frac{1}{2}$ $t = \frac{3}{2}$</td>
</tr>
</tbody>
</table>

Solution.
8 Miscellaneous

Problem A:

The following string of “less-equals” is relatively obvious:

\((-2) + (-2) + (\pm 1) \leq \{1 + (-2) \mid -1\} \leq \pm 1 \leq \{1 + (\pm 2) \mid -1\}\)

Determine which of these “less-equals” can be replaced by “=”, and which can be replaced by “<”.

Solution

\(\pm 1 - 2 = \{1 - 2 \mid -1\} < \pm 1 = \{1 + 2 \mid -1\}\).

Problem B:

Consider a strange new chess piece, called the “quarter-rook / semi-bishop”. Left can move this piece any number of squares WESTWARD; Right can move it any number of squares Southwest OR Southeast. Two such pieces inhabit different squares on the following board:

![Chess board diagram]

In one version of the game, the 2 pieces are independent and can jump over each other or land on the same square. In another version, jumping over is illegal and landing on the same square annihilates both. Who can win, as a function of starting positions?
9 Tiny-Second, et al

If \( x \) is a nonnegative integer, we may define:

\[
\begin{align*}
\pm_0^x &= x|0 ; & -^0_x &= 0|-x \\
\pm_1^x &= 0|-^0_x ; & -^1_x &= \pm_0^0|0 \\
\pm_2^x &= 0|-^0_x -^1_x ; & -^1_x &= \pm_0^0 +^1_0|0 \\
\pm_3^x &= 0|-^0_x -^1_x -^2_x ; & -^1_x &= \pm_0^0 +^1_0 +^2_0|0 \\
\vdots \\
\end{align*}
\]

If \( \{c_n\} \) is a finite sequence of integers, consider the game

\[ G = \sum_{n} c_n +^n_x. \]

For each case shown below, state whether \( G \) is positive, negative, zero or fuzzy. (If the cases need to be split into further subcases, do so.)

<table>
<thead>
<tr>
<th>( x ) is positive and ( c_0 \geq 2 )</th>
<th>then ( G ) is?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_0 = 1 = c_1 = c_2 = \cdots = c_{m-1} &lt; c_m )</td>
<td>( c_0 = 1 = c_1 = c_2 = \cdots = c_{m-1} &lt; c_m )</td>
</tr>
<tr>
<td>( c_0 = 1 = c_1 = c_2 = \cdots = c_{m-1} &gt; c_m )</td>
<td>( c_0 = 1 = c_1 = c_2 = \cdots = c_{m-1} &lt; c_m )</td>
</tr>
<tr>
<td>( c_0 = 0 = c_1 = c_2 = \cdots = c_{m-1} &lt; c_m )</td>
<td>( c_0 = 0 = c_1 = c_2 = \cdots = c_{m-1} &gt; c_m )</td>
</tr>
<tr>
<td>( c_0 = 0 = c_1 = c_2 = \cdots = c_{m-1} &gt; c_m )</td>
<td>( c_0 = 0 = c_1 = c_2 = \cdots = c_{m-1} &gt; c_m )</td>
</tr>
</tbody>
</table>

| \( x \) is zero and ... |
|-----------------|-----------------|
| \( c_0 \) is even | \( c_0 = 0 = c_2 = \cdots = c_{m-1} < c_m \) |
| \( c_0 = 0 = c_2 = \cdots = c_{m-1} < c_m \) | \( c_0 = 0 = c_2 = \cdots = c_{m-1} < c_m \) |
| \( c_0 = 0 = c_2 = \cdots = c_{m-1} > c_m \) | \( c_0 = 0 = c_2 = \cdots = c_{m-1} > c_m \) |

<table>
<thead>
<tr>
<th>( c_0 ) is odd</th>
<th>( c_0 = 1 = c_2 = \cdots = c_{m-1} &lt; c_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1 = 1 = c_2 = \cdots = c_{m-1} &lt; c_m )</td>
<td>( c_1 = 1 = c_2 = \cdots = c_{m-1} &lt; c_m )</td>
</tr>
<tr>
<td>( c_1 = 1 = c_2 = \cdots = c_{m-1} &gt; c_m )</td>
<td>( c_1 = 0 = c_2 = \cdots )</td>
</tr>
</tbody>
</table>
### Solution

<table>
<thead>
<tr>
<th>( x ) is positive and</th>
<th>then ( G ) is?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_0 \geq 2 )</td>
<td>positive</td>
</tr>
<tr>
<td>( c_0 = 1 = c_1 = c_2 = \cdots = c_{m-1} &lt; c_m )</td>
<td>positive</td>
</tr>
<tr>
<td>( c_0 = 1 = c_1 = c_2 = \cdots = c_{m-1} &gt; c_m )</td>
<td>fuzzy</td>
</tr>
<tr>
<td>( c_0 = 0 = c_1 = c_2 = \cdots = c_{m-1} &lt; c_m )</td>
<td>positive</td>
</tr>
<tr>
<td>( c_0 = 0 = c_1 = \cdots )</td>
<td>zero</td>
</tr>
<tr>
<td>( c_0 = 0 = c_1 = c_2 = \cdots = c_{m-1} &gt; c_m )</td>
<td>negative</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x ) is zero and ( \ldots )</th>
<th>( c_0 ) is even</th>
<th>( c_1 = 0 = c_2 = \cdots = c_{m-1} &lt; c_m ) positive</th>
<th>( c_1 = 0 = c_2 = \cdots = c_{m-1} &lt; c_m ) zero</th>
<th>( c_1 = 0 = c_2 = \cdots = c_{m-1} &gt; c_m ) negative</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( c_0 ) is odd</td>
<td>( c_1 \geq 2 ) positive</td>
<td>( c_1 = 1 = c_2 = \cdots = c_{m-1} &lt; c_m ) positive</td>
<td>( c_1 = 1 = c_2 = \cdots = c_{m-1} &gt; c_m ) fuzzy</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( c_1 = 0 = c_2 = \ldots ) fuzzy</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
10 Incentives

Problem. Prove or disprove the following. Let $G$ be a game in canonical form, and let $G^L$ be any left option. Then,

1. Either

$$G^L - G > -2^{-n}, \forall n$$

or

$$G^L - G = -2^{-k}, \text{ for some } k.$$

2. In the latter case, $G^L$ is the ONLY Left option of $G$ and $G$ is a number.

Solutions.

1. Let $m$ be the smallest nonnegative integer such that

$$G^L - G + 2^{-n} \neq 0.$$  

Then either (a) $G^{LR} - G + 2^{-n} \leq 0$, or (b) $G^L - G^L' + 2^{-n} \leq 0$ or (c) $n > 0$ and $G^L - G + 2(2^{-n}) \leq 0$.

If equality occurs in any case, then we have determined a value of $k$ such that $G^L - G = -2^{-k}$.

If not, then in (a) we have

$$G^{LR} \leq G - 2^{-n} < G,$$

contradicting the canonical form. Similarly, for (b), we have

$$G^L < G^L - 2^{-n} \leq G^L',$$

again contradicting the canonical form. In (c), we have contradiction of the minimality assumption of $n$.

2. (a) If $G^L - G = -2^{-k}$ and $G^{L'} - G > -2^{-k}$ then $G^{L'} > G^L$ so $G^L$ is a dominated option. Hence, if $G^L$ and $G^{L'}$ are any two canonical options, then

$$G^{L'} = G - 2^{-k} = G^L.$$

(b) We use confusion intervals:

$$G^L \quad G \quad G^R$$

If $G$ is not a number, then

$$L(G) = \max_{G^L} R(G^L)$$

and if $G^L = G - 2^{-k}$, then

$$L(G) = R(G) - 2^{-k} < R(G)$$

and this is a contradiction, as every game must have $L(G) \geq R(G)$. 

23
Problem. A game is all-small iff all its stopping positions are 0, or equivalently, iff it is either 0 or has both $G^L$ and $G^R$ both nonempty and all-small. Find a sequence of positive all-small games $g_1, g_2, \ldots, g_n, \ldots$ such that any positive all-small game born on day $n$ exceeds $g_n$. (Of course, $g_n$ will have to be born after day $n$.)

Solution (one of many adequate constructions: ) Let $g_n = \{0 \mid 0 \downarrow (n + 3) \uparrow\}$.
11 Paintenbush

**Problem.** Consider the following tree, in which the trunk is Blue and the other three branches are uncolored:

![Tree Diagram]

A: For EACH of the possible ways of coloring \(a, b, c\) with Blue/Red, determine the numerical value of the 4-branch LR Hackenbush tree at which painting stops.

B: What are the values of \(G^L_a\) and \(G^R_a\)? (assuming only branch \(a\) is colored)

C: Determine mean values, temperatures, and infinitesimal value at freezing of \(G^L_a\), \(G^R_a\) and \(G\).

**Solutions.**

(A)

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>L</td>
<td>L</td>
<td>+4</td>
</tr>
<tr>
<td>L</td>
<td>L</td>
<td>R</td>
<td>+2\frac{1}{2}</td>
</tr>
<tr>
<td>L</td>
<td>R</td>
<td>L</td>
<td>+2</td>
</tr>
<tr>
<td>L</td>
<td>R</td>
<td>R</td>
<td>+2\frac{1}{4}</td>
</tr>
<tr>
<td>R</td>
<td>L</td>
<td>L</td>
<td>+1\frac{1}{2}</td>
</tr>
<tr>
<td>R</td>
<td>L</td>
<td>R</td>
<td>+1\frac{1}{4}</td>
</tr>
<tr>
<td>R</td>
<td>R</td>
<td>L</td>
<td>+1\frac{1}{8}</td>
</tr>
<tr>
<td>R</td>
<td>R</td>
<td>R</td>
<td>+1\frac{1}{16}</td>
</tr>
</tbody>
</table>

(B)

\[ G^L_a = 4 \]
\[ G^R_a = 1\frac{1}{2} \]

(C)

\[ G^L_{5/8} = G^L_{5/8} = 2\frac{3}{4} \]
\[ G^L_{5/8} = 2\frac{1}{2} \]
\[ G^L_{3/4} = 2\frac{1}{2} * \]
\[ G^L_{15/16} = 2\frac{5}{16} * \]
\[
\mu(G^L) = 2 \frac{5}{16}, \quad t(G^L) = \frac{15}{16}
\]

\[
G_{1/8}^R = G_{1/8}^{R_b} = \begin{array}{c}
\frac{1}{4} \\
\frac{1}{2}
\end{array} \begin{array}{c}
\frac{3}{8} \\
\frac{7}{8}
\end{array} \quad G_{1/4}^R = 1 \begin{array}{c}
\frac{1}{2}
\end{array} \begin{array}{c}
\frac{1}{2}
\end{array}
\]

\[
G_{1/4}^R = \frac{1}{2} - \frac{1}{2}; \quad \mu(G^R) = \frac{1}{2}; \quad t(G^R) = \frac{1}{4}
\]

Finally,

\[
G_{3/4} = G_{3/4}^L - \frac{3}{4} \begin{array}{c}
G_{3/4}^R + \frac{3}{4}
\end{array}
\]

\[
= 1 \frac{3}{4} * \begin{array}{c}
\frac{3}{8}
\end{array} \begin{array}{c}
\frac{3}{4}
\end{array}
\]

\[
G_{7/8} = 1 \frac{1}{2} \begin{array}{c}
\frac{3}{8}
\end{array} \begin{array}{c}
\frac{3}{8}
\end{array} = 1 \frac{3}{8} \frac{1}{8}
\]

\[
\mu(G) = \frac{3}{8}, \quad t(G) = \frac{7}{8}
\]
12 Immortal Rooks

Problem. A loopy game may belong to any of 9 outcome classes, as follows:

<table>
<thead>
<tr>
<th></th>
<th>L starts</th>
<th>R starts</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L can win</td>
<td>Draw</td>
</tr>
<tr>
<td>R can win</td>
<td>Fuzzy</td>
<td>Draw</td>
</tr>
<tr>
<td></td>
<td>R can win</td>
<td>Draw!</td>
</tr>
<tr>
<td>L can win</td>
<td>Positive</td>
<td>0</td>
</tr>
</tbody>
</table>

Consider the game of immortal rooks on this 5-square L-shaped board.

For each possible starting position, determine the outcome class.

Solution.
13 Canonical Form and Mean Value

Canonical Form Problems:

Recall these definitions: $\ast = 0|0, \ast 2 = \{0, \ast | 0, \ast\}$

$\uparrow [1] = 0|\ast$
$\uparrow [2] = \uparrow [1]|\ast$
$\uparrow [3] = \uparrow [2]|\ast$

$\uparrow 2 = 0| \downarrow \ast, \quad \downarrow 2 = \uparrow \ast|0$
$\uparrow 3 = 0| \downarrow \uparrow 2 \ast, \quad \downarrow 3 = \uparrow \uparrow 2 \ast|0$

1. What is the canonical form of $\uparrow [n]\ast$?

2. What is the canonical form of $\uparrow [n] \ast 2$?

Solutions

$\uparrow [n]\ast = \{\uparrow [n - 1]\ast, \uparrow [n] | 0, \uparrow [n]\}$

$= \{\uparrow [n - 1]\ast, \uparrow [n] | 0\}$, because 0 dominates $\uparrow [n]\ast$

$= \{\uparrow [n - 1]\ast, 0 | 0\}$, because $(\uparrow [n])^R = \ast < \uparrow [n]\ast$

canonical

$\uparrow [n] \ast 2 = \{\uparrow [n - 1] \ast 2, \uparrow [n]\ast, \uparrow [n] | \ast 3, \uparrow [n]\ast, \uparrow [n]\}$

$= \{\uparrow [n - 1] \ast 2, \uparrow [n]\ast, \uparrow [n] | \ast 3\}$, other $G^R$ dominated

Now, $\uparrow [n]\ast$ reverses thru 0 to empty and $\uparrow [n]$ reverses thru $\ast$ to 0.

$\uparrow [n] \ast 2 = \{0, \uparrow [n - 1] \ast 2 | \ast 3\}$

$= \begin{cases} 
\{\uparrow [n - 1] \ast 2 | \ast 3\}, & \text{if } n > 1 \\
\{0 | \ast 3\}, & \text{if } n = 1 
\end{cases}$
Canonical Form Problem

Express the game ↑ + *2 in canonical form.

Solution

\[ \uparrow + * 2 = \{ \uparrow, \uparrow + *, *2 \mid *3, \uparrow, \uparrow + * \} \]

But \( \uparrow > *2 \) and \( *3 < \uparrow \), and \( *3 < \uparrow + * \), so

\[ \uparrow + * 2 = \{ \uparrow, \uparrow + * \mid *3 \} \]

But Right can move from \( \uparrow \) to \( * \) and \( * < \uparrow + * 2 \), so Left’s move from \( \uparrow + * 2 \) to \( \uparrow \) reverses through \( * \) to 0, and

\[ \uparrow + * 2 = \{ 0, \uparrow + * \mid *3 \} \]

Similarly, Left’s move from \( \uparrow + * 2 \) to \( \uparrow + * \) reverses through 0 to empty, so

\[ \uparrow + * 2 = \{ 0 \mid *3 \} \]

is the canonical form.
Tinies, Minies and Canonical Forms

If $x$ is any nonnegative number, then miny-$x$ is defined as:

$$-x = x \mid 0 \parallel 0$$

(a) Let $n$ be an integer greater than 1. Express $n \cdot (-)_x$, the sum of $n$ copies of miny-$x$, in the form

$$n \cdot (-)_x = \{G^{LL} \mid G^{LR} \parallel G^R\}$$

where each of $G^{LL}$, $G^{LR}$, $G^R$ is a minimal set of sums of $x$, 0, tinies, minies, and numbers, and each such sum is simplified as much as possible.

(b) Check your answer to (a) by setting $x = 0$ to obtain an expression for $n \cdot \downarrow$. 
Problems on Canonical Forms (possibly wrong conjecture??)

Define

\[ \epsilon_0 = 0 \quad \|\left(-\frac{\varphi}{4} + \int \ast + \int \int 2^{-0}\right) = +_{1\frac{1}{2}} \]
\[ \epsilon_i = 0 \quad \|\left(-\frac{\varphi}{4} + \int \ast + \int \int 2^{-i}\right) \]

where \( \int \ast \) means \( \int \frac{3}{2} \ast \) and \( \int \int \) means \( \int \frac{3}{2} \int \frac{3}{2} \ast \). So \( \epsilon_1 = +_2 \) and \( \epsilon_n > \epsilon_{n+1} > \cdots > +_{2\frac{1}{2}} > 0 \).

(a) Prove that \( \epsilon_n >> \epsilon_{n+1} \).

(b) Reduce the following to canonical forms:

\[
\begin{align*}
-\frac{3}{4} + \int \ast + \int \int 1\frac{1}{2} - 2\epsilon_1 \\
-1 + \int \int 1\frac{1}{4} - 2\epsilon_1 - \epsilon_2 \\
-1\frac{1}{4} + \int \ast + \int \int 1\frac{7}{8} - 2\epsilon_1 - \epsilon_2 - \epsilon_3 \\
-1\frac{1}{2} + \int \int 2 - 2\epsilon_1 - \epsilon_2 - 2\epsilon_3 \\
-1\frac{3}{4} + \int \ast + \int \int 2\frac{1}{4} - 2\epsilon_1 - 2\epsilon_2 - 2\epsilon_3
\end{align*}
\]
Canonical Forms

**Problem.** Express each of the following 8 games in canonical form. A thru D are Domineering positions; E thru H are abstract games. Recall that in domineering, Left plays vertical; Right, horizontal.

\[ A = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \end{array} \]
\[ B = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \end{array} \]
\[ C = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \end{array} \]
\[ D = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \end{array} \]

\[ E = \uparrow * \]
\[ F = \text{the sum of these three games } \{x|0\} + x + x \ (\text{recall } +_x = \{0||0| - x\}) \]
\[ G = \{\uparrow | -2, *\} \ (\text{recall } -_2 = \{2|0||0\}) \]
\[ H = \downarrow _2*, \text{ the sum of down-second and star } (\text{recall } \uparrow^2 = \{0| \downarrow *\}) \]

**Solutions.**

\[ A = -\frac{1}{2} = 1|0 \]
\[ B = \downarrow = *|0 \]
\[ C = 0 \]
\[ D = -_2 = 2|0||0 \]
\[ E = \uparrow * = 0|\uparrow \]

\[ F = \{x|0\} + x + x \ (\text{reverses}) \]
\[ G = \{\uparrow | -_2, *\} = \{1| 0, *\} \]
\[ H = \{\downarrow | 0\} + * = \uparrow |0,* \]
14 Prove or Disprove

Fundamentals

Decide whether each of the following statements is true.

1. If \( G = G^L \upharpoonright G^R \) and if there is any number \( x \) such that no \( G^L \geq x \) and no \( G^R \leq x \), then \( G \) is equal to the simplest such \( x \).

2. If each \( G^L \) is \( \leq \) some nimber, and if \( G^R = -G^L \), then \( G \) is the simplest nimber not in \( G^L \).

3. If Birthday(\( G \)) = \( n \), an integer, then \( G \leq n \).

4. If \( g = 0 \)-ish, and if Birthday(\( g \)) = \( n + 1 \), then \( g \leq n \cdot \uparrow \).

5. If \( G \) is less than all positive numbers and greater than all negative numbers, then \( G = 0 \).

6. If all \( G^L \) < all \( G^R \), then \( G \) is a number.

7. If \( G \) has no Right followers, then \( G \) is a nonnegative integer.

8. If \( G \) has no Right follower \( \leq 0 \), then \( G > 0 \).

9. If \( G \) is a nimber, and \( G \) is not equal to \( * \), then \( \downarrow < G < \uparrow \).

10. If \( t(G) \) is a positive number, then \( G \) cooled by \( t \) is a number.

11. If \( G \) is a game which becomes 0 when cooled by 1, then \( G = 0 \)-ish.

12. If \( G \) is a sum of Blockbusting games which is not equal to 0, but becomes 0 when cooled by 1, then \( G = * \).

13. If \( H \) is cold but \( G \) is not, and if Left has a winning move from \( G + H \), then he has a winning option of the form \( G^L + H \).

14. If \( G \) is hot and \( H \) is tepid, and if Left has a winning move from \( G + H \), then he has a winning option of the form \( G^L + H \).

15. If Left can win from \( G + \sum_i H_i \) and if \( G \) is a positive game with canonical form \( G = \{ 0 \mid G^R \} \), then Left has a winning move playing on some \( H_i \).

Solutions.

1. TRUE.

2. TRUE.

3. TRUE.
4. FALSE. Let \( n = 1 \) and \( g = \uparrow \ast \). Tight correction: if \( g = 0 \)-ish, and if \( \text{Birthday}(g) = n+1 \), then either \( g \leq n \cdot \uparrow \) or \( g \leq n \cdot \uparrow + \ast \). Note: is the weaker version really true? Take the above example of \( n = 1 \) and \( g = \uparrow \ast \ldots \)

5. FALSE. Let \( G = \ast \). Correction: if \( G \) is less than all positive numbers and greater than all negative numbers, then \( G \) is an infinitesimal and \( G = 0 \)-ish.

6. FALSE. Take \( 0 \uparrow = \uparrow \ast \). Correction: if all \( G^L < \) all \( G^R \), and if the dominant \( G^L \) and \( G^R \) are numbers, then \( G \) is the simplest number between \( G^L \) and \( G^R \).

7. TRUE.

8. FALSE. Let \( G = 0 \). Correction: if \( G \) has no Right follower \( \leq 0 \), then \( G \geq 0 \).

9. TRUE. Every nimber except \(*1\) is less than \( \uparrow \).

10. FALSE. The game \( 1|−1 \) has temperature 1, but when cooled by 1, it becomes \(*\). Correction: if \( t(G) \) is a positive number, then \( G \) cooled by \( t \) is number-ish.

11. FALSE. Counterexample: \( \frac{1}{2}|−\frac{1}{2} \) chills to 0. Correction: if \( G \) is a game which has only integer stops, and which becomes 0 when cooled by 1, then \( G = 0 \)-ish.

12. TRUE. This is a consequence of the parity principle. A similar and very important result holds for normal Go endgame positions.

13. TRUE. This is also known as the number avoidance theorem.

14. FALSE. From \( 1|−1 + −2 \), Left’s only winning option is to play on \(-2\). Correction: if \( G + H \) is hot, and if Left has a winning move from \( G + H \), then he has a winning move whose incentive is hot.

15. FALSE. From \( \uparrow + \ast + \ast \), Left’s only winning option is on \( \uparrow \). Correction: if \( G + \sum H_i \) is irreducible (i.e., no subset of summands sums to 0), and Left can win from \( G + \sum H_i \), and if \( G \) is a positive game with canonical form \( G = \{ 0 \mid G^R \} \), then Left has a winning move played on some \( H_i \).
Prove this theorem on Incentives

Let $G$ be a game in canonical form, and let $G^R$ be any Right option. Then:

1. Either $G - G^R > -2^{-n}$ for all $n = 0, 1, 2, \ldots$ or $G - G^R = -2^{-k}$ for some integer $k$.

2. In the latter case, $G^R$ is the only Right option of $G$ and $G$ is a number.

Proof: 1. Let $n$ be the smallest nonnegative integer such that $G^L - G + 2^{-n} \neq 0$. Then either:

(a) $G^{LR} - G + 2^{-n} \leq 0$, or
(b) $G^L - G^{L'} + 2^{-n} \leq 0$, or
(c) $n > 0$ and $G^L - G + 2(2^{-n}) \leq 0$.

If equality occurs in any case, then we have determined a value of $k$ such that $G^L - G = -2^{-k}$. If not, then in (a), we have

$$G^{LR} \leq G - 2^{-n} < G,$$

contradicting the fact that $G$ is in canonical form. Similarly in (b), we have

$$G^L < G^{L'} - 2^{-n} \leq G^{L'},$$

contradicting the fact that $G$ is in canonical form. In (c), we have a contradiction of the minimality assumption of $n$.

2a. If the following hold

$$G^L - G = -2^{-k}, \quad G^{L'} - G > 2^{-k}$$

then $G^{L'} > G^L$, so $G^L$ is a dominated option. Hence, if $G^L$ and $G^{L'}$ are any two canonical options, then

$$G^{L'} = G - 2^{-k} = G^L.$$

2b. We have confusion intervals

If $G$ is not a number, then

$$L(G) = \max_{G^L} R(G^L)$$

and if $G^L = G - 2^{-k}$, then

$$L(G) = R(G) - 2^{-k} < R(G)$$

and this is a contradiction, as every game must have

$$L(G) \geq R(G).$$
Prove or disprove each of the following assertions, assuming that all games are short.

1. The freezing temperature of $G + H$ is equal to the maximum of the freezing temperatures of $G$ and $H$.

2. If every right option of $G$ exceeds every left option of $G$, then $G$ is a number.

3. If every right option of $G$ exceeds every left option of $G$, then $G$ is the sum of a number and an infinitesimal.

4. If $G$ is not a number, and $G = \{A, B, C \mid X, Y, Z\}$ then

$$G + * = \{A + *, B + *, C + * \mid X + *, Y + *, Z + *\}.$$

Solutions.

1. False, $\pm 1$ has temperature 1 and $\pm 1 + \pm 1$ has temperature $0$.

2. False, $\{0|\uparrow\} = \uparrow* \text{ is not a number}$.

3. True. Let $L(G)$ be the Left end of the confusion interval of $G$; $R(G)$ be the Right end.
   If $\min L(G^R) < \max R(G^L)$, then $G$ is a number, so assume that
   $$\min L(G^R) = \max R(G^L) = x = \text{number},$$
   then $G$ is $x$-ish.
   Furthermore, $G - x$ has integer atomic weight, but we shall not prove that here.

4. False, $\uparrow = 0|* \text{ but } \uparrow +* = \{0,*,0\} \neq *0 = \downarrow$.
   Or $\star 2 = \{0,|0,*\}$ but $\star 3 = \star 2 + \star \neq \{*,0|*,0\} = \star 2$. 

36
Prove or disprove the following assertions, assuming all games are short.

1. If \( g \) is a game born on or before Day \( t + 1 \), and if \( g \) does not exceed any positive number, then either \( g = t \cdot \uparrow \) or
   \[
g < (t + 1) \cdot \uparrow + *\]

2. (Left out, as it was already covered.)

3. If for each \( g^R \) in \( G^R \), there exists some integer \( n \) such that \( n \cdot g^R \geq 0 \), then the game \( g = \{-G^R \mid G^R\} \) is equal to an impartial game.

4. If every right successor of \( g \) exceeds every left successor of \( g \), then \( g \) is the sum of a number and an entire infinitesimal.

5. (Hard) If \( g \) exceeds all of its left successors and \( g \) is less than all of its right successors then \( g \) is a number.

Solutions.

1. False unless \( t \leq 0 \). A counterexample is the game \( \pm 1 \). However, the assertion is true for small games.

2. (Left out)

3. True: first observe that no \( g^R < *k \); for if some \( g^R < *k \), then \( 2n \cdot g^R < 0 \), which contradicts the assumption that \( n \cdot g^R \geq 0 \). Therefore, if we define \( j \) as the integer for which \( 0, *, *2, \ldots, *(j - 1) \) are in \( G^R \) but \( *j \) is not, then \( g = *j \).

4. If \( \min L(g^R) < \max R(g^L) \), then \( g \) is a number, so we assume that \( \min L(g^R) = \max R(g^L) = x \) is a number. In view of the translation theorem, it remains to show only that \( h = g - x \) is infinitesimal, given that
   \[
   L(h) = R(h) = 0, \quad \text{and } h^L < h^R.
   \]
   If any \( l(h^R) + 1 \leq n(h^L) - 1 \), then we obtain a contradiction:
   \[
h^R < (l(h^R) + 1) \cdot \uparrow + * \infty < h^L
   \]
   so we deduce that
   \[
d(h) > \mu(h)
   \]
   and \( h \) is entire by Theorem ??.

5. As in the previous problem, everything reduces to the case when \( g \) is an infinitesimal, and then \( g = i \cdot \uparrow \) and \( \mu(g) < d(g) \). The corollary to theorem ?? gives us 3 cases:
   (a) \( \mu < i < d, \ g = *n \). Since \( *n \) has incomparable successors unless \( n = 0 \), we have \( g = 0 \).
(b) $\mu = i < d$. Now some $g^L$ has $R(g^L) = \mu + 1 = i + 1$, where we obtain the contradiction
\[ g^L \not< (i + 1) \cdot \uparrow + * \infty \]
even though $g < (i + 1) \cdot \uparrow + * \infty$ and $g^L < g$.
(c) $\mu < i = d$. Now some $g^R$ has $l(g^R) = i \neq 1$ and
\[ (i - 1) \cdot \uparrow + * \infty \not< g^R \]
even though $(i - 1) \cdot \uparrow + * \infty < g < g^R$, which is another contradiction.
Hence $g = 0$. 

38
Mean Value

Prove or disprove: if $a, b, c, d$ are numbers and $g$ is a game whose simplest form is

$$g = a \ || \ b \ || \ c \ || \ d$$

then its mean value is given by

$$g_\infty = \text{median} \{ \alpha, \frac{b+c}{2}, \frac{a+b+c}{2} \}$$

where $\alpha = \frac{a}{2} + \frac{b}{8} + \frac{c}{8} + \frac{d}{4}$.

Solution No solution attached in the file.
15 Heating

Let $G$ be a game whose canonical form is $G = \{G^L \mid G^R\}$. Recall that normal heating is defined as:

$$
\int^t G = \begin{cases} 
G, & \text{if } G \text{ is a number} \\
\{t + \int^t G^L \mid -t + \int^t G^R\}, & \text{otherwise}
\end{cases}
$$

Recall also that overheating is defined as:

$$
\int^b_a G = \begin{cases} 
a \cdot G, & \text{if } G \text{ is an integer} \\
\{b + \int^b_a G^L \mid -b + \int^b_a G^R\}, & \text{otherwise}
\end{cases}
$$

We no longer restrict $b$ to be a number; we now also allow number + *. Express each of the following games in canonical form, showing all stops explicitly.

(a) $-\frac{7}{8} + \int^{\frac{1}{8}}_{\frac{1}{8}}*$,
(b) $\int^{\frac{1}{4}}_{\frac{1}{2}} \int^{\frac{1}{8}}_{\frac{1}{2}}$,
(c) $\int^{\frac{1}{4}}_{\frac{3}{4}} \int^{\frac{3}{4}}_{\frac{3}{4}}$.

Note: Berlekamp has shown that expressions such as these solve infinitely many positions in $3 \times n$ Domineering, such as

$$
-3 + \int^{\frac{1}{4}}_{\frac{1}{4}} \int^{\frac{3}{4}}_{\frac{3}{4}} = \begin{array}{ccccccc}
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1}
\end{array}
$$

Solutions. (a) $\int^{\frac{1}{8}}_{\frac{1}{8}} \{0|0\} = 1\frac{1}{8} \mid -1\frac{1}{8}$. So

$$
-\frac{7}{8} + \int^{\frac{1}{8}}_{\frac{1}{8}}* = \frac{1}{4} \mid -2
$$

(b) $\int^{\frac{1}{2}}_{\frac{1}{2}} = \int^{\frac{1}{4}}_{\frac{1}{4}} \{0|1\} = \{1\ast \mid \frac{1}{4}\} - \frac{1}{8} = \frac{1}{4}\ast \mid *

$$
\int^{\frac{1}{4}}_{\frac{1}{4}} \int^{\frac{1}{2}}_{\frac{1}{2}} = 2\frac{1}{2} \mid \frac{1}{4} || 0 \mid -2\frac{1}{4}
$$

(c) $\int^{\frac{3}{4}}_{\frac{3}{4}} = \int^{\frac{3}{4}}_{\frac{3}{4}} 0 \mid 1 \mid 1 = \frac{1}{4} \mid \frac{1}{4} \mid \ast

$$
\int^{\frac{1}{8}}_{\frac{3}{4}} \int^{\frac{3}{4}}_{\frac{3}{4}} = 2\frac{3}{4} \mid \frac{1}{4} \mid 0 \mid -2\frac{1}{4}
$$
Heating Problem. Assume that $s$ is a positive number, and that $n$ is a number of the form $r = 2^{-n}$, where $n$ is a natural number (i.e. nonnegative integer).

1. Express the canonical form of $\int_r^s \frac{1}{2}$ and of $\int_r^s \frac{1}{4}$ in terms of slashes, numbers and stars.

2. Suppose $x$ is either a number or a number plus star. What is the temperature (i.e. freezing point) of $\int_r^s x$? Express the answer in terms of $r$, $s$ and incentive($x$).

Solutions.

1. $G = \int_r^s \frac{1}{2} = s + \int 0 \mid -s + \int 1 .

   G = s \mid r - s = \frac{r}{2} \pm (s - \frac{r}{2})$.

   $G_{s - \frac{r}{2}} = \frac{r}{2} \mid \frac{r}{2} = \frac{r}{2} *$ so the temperature is $s - \frac{r}{2} . H = \int_r^s \frac{1}{4} = s + \int 0 \mid -s + \int 1 \frac{1}{2} .

   H = s \mid 0 \mid r - 2s$

   $H_{s - \frac{r}{4}} = \frac{r}{4} \mid 0 \mid \frac{r}{2} . \text{ Since } r = 2^{-n}, \frac{r}{2} = 2^{-n-1} \text{ and } 0|\frac{r}{2} = 2^{-n-2} . \text{ Hence }$

   $H_{s - \frac{r}{4}} = \frac{r}{4} *$

   so the temperature is $s - \frac{r}{4}$.

2. We assert that:

   Temperature $= \begin{cases} 0, & \text{if } x \text{ is an integer,} \\ s, & \text{if } x \text{ is a number plus } *, \\ s + r \cdot \text{Incentive}(x), & \text{if } x \text{ is a non-integer number.} \end{cases}$

First, the temperature of $\int *$ is obviously $s$. The temperature of $\int n = n \cdot r$ is obviously $0$.

If $x$ is a number that is not an integer then $x = x_L \mid x_R$. Define $d_x = -\text{Incentive}(x)$, so $d_x = 2^{-k}$. Then

$x_L = x - 2^{-k}, \quad x_R = x + 2^{-k}$

By problem 1, the above assertion is true if $x$ is half-integer. As an induction hypothesis, we assume that

\[ \left( \int_r^s x_L \right)_{s-2r-d_x} \overset{\text{ish}}{=} r \cdot x_L = r \cdot (x - d_x) \]

\[ \left( \int_r^s x_R \right)_{s-2r-d_x} \overset{\text{ish}}{=} r \cdot x_R = r \cdot (x + d_x) \]

whence

\[ \left( \int_r^s x \right)_{s-2r-d_x} = \left\{ s + \int x_L \mid -s + \int x_R \right\}_{s-2r-d_x} \]

\[ = \left\{ 2r \cdot d_x + r \cdot x_L \mid -2r \cdot d_x + r \cdot x_R \right\} \]

\[ = r \cdot x + \left\{ r \cdot d_x \mid -r \cdot d_x \right\} \]

41
and
\[
\left( \int_{r}^{s} x \right)_{s-r \cdot d,s} = r \cdot x + i_{sh} r \cdot x
\]

So if (as presumed by induction hypothesis) \((\int_{r}^{s} x^{L})\) and \((\int_{r}^{s} x^{R})\) freeze at \(s - 2 \cdot r \cdot d_{x}\) or lower, then \(\int_{r}^{s} x\) freezes at \(s - r \cdot d_{x}\).
16 Domineering

For each of the following Domineering positions, determine the canonical form, the Left incentive, and the Right incentive.

If \( F = A + B - C - D - E \), can Left moving first win from \( F \)?

Solutions.

\[
\begin{align*}
A &= 1, & \Delta^L A = -1, & \Delta^R A = \text{empty} \\
B &= 1|-1, & \Delta^L B = \Delta^R B = 2|0 \\
C &= 2|-\frac{1}{2}, & \Delta^L C = \Delta^R C = 2\frac{1}{2}|0 \\
D &= 2|0||0, & \Delta^L D = \{2|0\} + [2], & \Delta^R D = -2 \\
E &= 2 + \frac{1}{2} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \Delta^L E = \Delta^R E = -\frac{1}{2}
\end{align*}
\]

\[
F = A + B - C - D - E
= 1 + \{1|-1\} + \{\frac{1}{2}|-2\} + 2 - \frac{1}{2},
\]

and the dominant canonical incentive is \( \Delta C \). So if Left plays first, he plays sum to \( 1 + \{1|-1\} + \{\frac{1}{2}|-2\} + 2 - \frac{1}{2} = F^L \).

Right’s dominant move on \( F^L \) is on \( +_2 \), to \( 0|-2 \), so

\[
F^{LR} = 1 + \{1|-1\} + \{0|-2\} = 0.
\]

Hence Left first cannot win. But Right first can, so

\[
A + B - C - D - E < 0.
\]

Note that from \( F^L \), the noncanonical Right move to \( (-E)^R = -A - C \) also wins.
Domineering Problems by Jim Borger (Spring 1995)

(A) Who (Left, Right, 1st or 2nd) wins the following domineering game?

(B) What if it is cooled by $\frac{1}{2}$?

Solutions. (A) We have

$$G = \pm 1 - \frac{1}{2} - 1 + 0 - 1 + 3 \mid 1 \frac{1}{2}$$

The hottest game is $\pm 1$, followed by $3 \mid 1 \frac{1}{2}$ and $0 - 1$. Hence Left first stops at

$$1 + 1 \frac{1}{2} + 0 + (-1 \frac{1}{2}) = 1 > 0.$$ 

Right first stops at

$$-1 + 3 - 1 + (-1 \frac{1}{2}) = -\frac{1}{2} < 0.$$ 

So the first player wins $G$.

(B) We have

$$G_{\frac{1}{2}} = \frac{1}{2} \mid -\frac{1}{2} - 1 \frac{1}{2} - \frac{1}{2} * + 2 \frac{1}{2} \mid 2.$$ 

The hottest game is $\frac{1}{2} \mid -\frac{1}{2}$, followed by $2 \frac{1}{2} \mid 2$ and $-\frac{1}{2} *$. Hence Left first stops at

$$\frac{1}{2} + 2 - \frac{1}{2} - 1 \frac{1}{2} = \frac{1}{2} > 0.$$ 

Right first stops at

$$-\frac{1}{2} + 2 \frac{1}{2} - \frac{1}{2} - 1 \frac{1}{2} = 0,$$

and Right gets the last move. Hence the first player wins $G_{\frac{1}{2}}$. 

44
Some Domineering Positions

Evaluate each of the following Domineering positions.

Solution

What if we proceed onward in each column?

The solution appears in Winning Ways.
Domineering Problem. The subsequences of the following Domineering position have the values shown:

\[
\begin{array}{|c|c|}
\hline
A & B \\
\hline
C & D \\
\hline
E & F \\
\hline
G & H \\
\hline
I & \\
\hline
\end{array}
\]

\[
\begin{align*}
AA &= 0 & AE &= 0 & DI &= \frac{1}{2} \ast & HI &= \uparrow \\
AB &= -1 & AF &= \frac{1}{2} \ast & EI &= 1 \ast & II &= 0 \\
AC &= \ast & AG &= 1 & FI &= 1 \mid 0 \\
AD &= \frac{1}{2} & CI &= 1 & GI &= \ast
\end{align*}
\]

1. Using these values, find the value of AI. (That is, the full position shown above.)

2. Determine the mean and temperature of AI.

3. Find \( s, t \) and a number \( x \), such that

\[
AI = \ast + \int_s^t x
\]

(Hint: \( x\mu(s) \) is the mean.)

4. Except for a possible additive infinitesimal (of value \( \ast \)), can you also express DI and FI in terms of this same overheating operator?

Solutions

1. 

\[
AI = \begin{cases} 
0 + \frac{1}{2} \ast, & 0 + 1, \\
-1 + 1 \ast, & 0 + 1, \\
\frac{1}{2} + \ast, & * + 1 | 0, \\
0 + 1, & \frac{1}{2} \ast + 0 \\
1 + 0 & \end{cases} = 1 \mid \frac{1}{2} \ast
\]

For the values of the other subsequences, we have

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& H & -1 & \\
\hline
G & +1 & \ast & \\
\hline
F & -1 & * & 0 | -1 & \\
\hline
E & -1 & -1 & -\frac{1}{2} & -1 \ast & \\
\hline
D & +1 & \ast & -\frac{1}{2} & 0 & * - \frac{1}{2} & \\
\hline
C & -1 & * & 0 | -1 & -1 \ast & * - \frac{1}{2} & -1 & \\
\hline
B & -1 & -1 & -\frac{1}{2} & -1 \ast & -1 \frac{1}{2} & -1 & -1 & \\
\hline
A & +1 & * & -\frac{1}{2} & 0 & * - \frac{1}{2} & -1 & * & \\
\hline
\end{array}
\]
2. $AI_{\frac{1}{4}} = \frac{3}{4} |\frac{1}{4}| \frac{1}{4}$. So $AI_{\frac{1}{4}+\frac{1}{8}} = \frac{5}{8}$; $\mu(AI) = \frac{5}{8}$; $t(AI) = \frac{3}{8}$.

3. 

\[
AI + * = 1 * \left|\left|\frac{1}{2}\right|\left|0\right|\right\} = \frac{1}{2} * + \left\{ \left|\frac{1}{2}\right| \left|0\right| - \frac{1}{2} * \right\} = \int_{\frac{1}{2} *}^{\frac{5}{4}} = \int_{\frac{1}{2} *}^{\frac{3}{4}} 1 + \int_{\frac{1}{2} *}^{\frac{1}{4}}.
\]

4. 

\[
DI = \frac{1}{2} * = \int_{\frac{1}{2} *}^{\frac{1}{2}} 0 = \int_{\frac{1}{2} *}^{\frac{1}{2}} 1
\]

\[
FI* = 1 * = \frac{1}{2} * + \int_{\frac{1}{2} *}^{\frac{1}{2}} * = \int_{\frac{1}{2} *}^{\frac{1}{2}} 1 *
\]
2 × n Domineering Problem

You are to play a Domineering game which is the sum of four pieces:

```
  x
  x
  x
  x
```

Begin your analysis by applying problems 2D and 2F to the sum of the first three pieces. Then decide which ONE of the four choices you prefer: {Left, Right, first, second}.

**Solution.** First we show that the first three games sum to 1|(-1 + 2). Indeed, we have

```
F = {x|0} + x = x|x
{x|0} + x
{0} (reverses)
{x|0} (reverses)
{x} (reverses)
```

and in particular \(2|0 + 2 + 2 = 2| + 2\). Subtracting 1 gives

\[
\{1| - 1\} + 2 + 2 = 1|(-1 + 2).
\]

Hence,

\[
\begin{array}{c}
2 \times 6 = 2 \times 5 , 1-2 , \pm 1 + 2 \mid \frac{1}{2} \mid \{ \pm 1 \mid -1 | -3 , -1 \} , \pm 1 , -1 \\
= \frac{1}{2} , 1-2 , \frac{3}{4} \pm 1 \frac{1}{4} \pm 1 \mid \text{same as previous line} \\
= 1 -2 \mid -1
\end{array}
\]

So the sum of the games is 0 and you should play second.
Show that the following Domineering positions have the canonical forms as claimed:

\[
\begin{align*}
\begin{array}{|c|c|}
\hline
\hline
\end{array} &= \pm 1 \frac{1}{2} \\
\begin{array}{|c|c|c|}
\hline
\hline
\end{array} &= -\frac{3}{8} \pm \frac{13}{8} \\
\begin{array}{|c|c|c|}
\hline
\hline
\end{array} &= -4 \frac{7}{8} \pm \frac{9}{8} \\
\begin{array}{|c|c|c|c|c|}
\hline
\hline
\end{array} &= -\frac{1}{2} - 3 \left| -3\frac{1}{2} - 5\frac{1}{2} \right| - 5\frac{3}{4}
\end{align*}
\]
**Problem.** Using Wolfe’s Gamesman Toolkit or Siegel’s CGSnite or any other methods you wish, determine which of the following four conjectured values is incorrect.

\[
\begin{align*}
\begin{array}{|c||c|c|}
\hline
\cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot \\
\hline
\end{array} & = \frac{1}{2} \bigg| -\frac{1}{2} \bigg| - 2 \\
\begin{array}{|c||c|c|}
\hline
\cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot \\
\hline
\end{array} & = 0 \bigg| -\frac{1}{2} \bigg| - 2\frac{1}{2} \\
\begin{array}{|c||c|c|}
\hline
\cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot \\
\hline
\end{array} & = 0 \bigg| -\frac{1}{2} \bigg| - 2\frac{1}{2} \\
\begin{array}{|c||c|c|}
\hline
\cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot \\
\hline
\end{array} & = -1
\end{align*}
\]

**Solution.** They are all correct.
17 Thermographs and Confusion Intervals

Thermograph Problems

\[ A = 10 \begin{array}{c|c|c|c} \hline 1 & 0 & -2 & -20 \hline \end{array} \]
\[ B = 13 \begin{array}{c|c|c|c} \hline 11 & 10 & -8 & -10 \hline \end{array} \]

1. Draw the thermographs of \( A \) and of \( B \). For each game, determine its mean, its temperature, and the infinitesimal obtained when the game minus its mean is cooled by its temperature.

2. Consider the game \( G = A + 5 \cdot B \).
   - What is the mean of \( G \)?
   - What is the temperature of \( G \)?
   - What is the infinitesimal obtained when \( G \) minus its mean is cooled by its temperature?

Each player seeks to optimize the stopping value of a game played, starting from \( G \).

- If Left plays first, what is the best option, \( G^L \)?
- If Right plays first, what is the best option, \( G^R \)?

Solutions.

1.

\[ \begin{align*}
\mu(A) &= 0, \\
t(A) &= 10, \\
A_{10} &= \uparrow,
\end{align*} \quad \begin{align*}
\mu(B) &= 1, \\
t(B) &= 10, \\
B_{10} - 1 &= *
\end{align*} \]
2. We have the following values:

\[
\begin{align*}
\mu(G) &= \mu(A) + 5\mu(B) = 5, \\
t(G) &= \max\{t(A), t(B)\} = 10, \\
G_{10} - 5 &= \uparrow *
\end{align*}
\]

From \(\uparrow *\), Left plays * to 0.
From \(G\), Left plays to \(G^L = A + B^L + 4B\).

From \(\uparrow *\), Right plays \(\uparrow\) to *.
From \(G\), Right plays to \(G^R = A^R + 5B\).
Thermograph Problem by Jim Borger (Spring 1995)

1. Find a game with the following thermograph:

![Thermograph Diagram]

2. How does any game $G$ with the above thermograph relate to 0? (Be as specific as possible.)

Solutions.

1. Take $(7|3), 4 \parallel 0|\neg2$, so that

![Solutions Diagram]

2. $G \parallel 0$ or $G > 0$. For example, $(7|3), 4 \parallel 0|\neg2 > 0$, but $(7|3), 4 \parallel *|-2 \not\in 0$.

Note: I can’t find the symbol for ”confused or greater”.

53
5-Part Problem

1a,b,c : Find the thermograph, the mean, and the temperature of each of the following games

\[
\begin{align*}
A &= \frac{1}{4} \left| -1 \right| -2 \\
B &= \frac{1}{4} \left| 0 \right| -3 \\
C &= \frac{1}{4} \left| 1 \right| 0 \left| -1 \right|
\end{align*}
\]

1d : What is the mean value and temperature of

\[
D = 5 \cdot A + 10 \cdot B + 15 \cdot C
\]

1e : Derive the tightest bounds you can find on \text{Leftstop}(D) and \text{Rightstop}(D). The more accurate your answer, the more credit you will receive.

Solution to 5-Part Problem

1a,b,c:

\[
\begin{align*}
t(A) &= \frac{5}{4} \\
\mu(A) &= -\frac{1}{4}
\end{align*}
\]

\[
\begin{align*}
t(B) &= 1 \\
\mu(B) &= 0
\end{align*}
\]
\[ t(C) = \frac{9}{8} \]
\[ \mu(C) = +\frac{1}{8}. \]

1d: We have
\[ \mu(D) = 5 \left( -\frac{1}{4} \right) + 10 \cdot 0 + 15 \left( \frac{1}{8} \right) = +\frac{5}{8} \]
\[ t(D) \leq \max \left\{ \frac{5}{4}, 1, \frac{9}{8} \right\} = \frac{5}{4} \]
and \( t(D) = \frac{5}{4} \), because \( D_{\frac{5}{4}} = * \neq 0 \).

1e: We have \( \frac{5}{8} + \frac{5}{4} = 1\frac{7}{8} \geq \text{Leftstop}(D) \geq \frac{5}{8} \geq \text{Rightstop}(D) \geq -\frac{5}{8} \). But the stops of \( D \) are integers, whence \( \text{Leftstop}(D) = 1 \) and \( \text{Rightstop}(D) = 0 \).
**Thermograph Problem**

$X$ and $Y$ are games with these canonical forms:

\[
X = \{a|b, \ c|d, \ || \ e|f\}
\]

\[
Y = \{g \ || \ h|i \ || \ j|k|l\}
\]

where $a, b, c, \ldots, l$ are numbers, and $a > c$. $X$ and $Y$ have the same thermographs:

2a. Give at least one set of possible values for the numbers $a, b, c, \ldots, l$.

2b. Give ranges for any of the numbers which can assume more than one value.

**Solution for Thermograph Problem**

2a

\[
X = 11|1, \ 6|2 \ \parallel \ \ 0|f; \quad f \leq -12 \text{ (possibly } f = -12)\
\]
\[ Y = 11 \mid 6 \mid 2 \mid -2 \mid 0 \mid l; \quad l \leq -12 \text{ (possibly } l = -12) \]

2b. The values as shown above are unique, except for

\[-12 \geq l > -\infty, \quad -12 \geq f > -\infty\]
More Thermograph Problems

5a,b. Find the thermograph, the mean and the temperature of each of the following games:

\[
A = 10 \mid 5 \mid 4 \mid 2
\]
\[
B = 2 \mid 0 \mid -1 \mid -2 \mid -3 \mid -4 \mid -5
\]

5c. What is the mean value and temperature of

\[
C = 7 \cdot A + 23 \cdot B
\]

5d. Derive the tightest bounds you can find on \( \text{Leftstop}(C) \) and \( \text{Rightstop}(C) \). (The more accurate your answer, the more credit you will receive.

Solutions.
5a,b. The thermographs of \( A \) and \( B \) give:

For small \( t \),
\[
A_t = 10 - 3t \mid 5 - t \mid 4 \mid 2 + t
\]
\[
A_1 = 7 \mid 4 \mid 3 = 4 - 3 \mid 3
\]
\[
A_{1\frac{1}{2}} = 3\frac{1}{2} \ast; \quad \mu(A) = 3\frac{1}{2}; \quad t(A) = 1\frac{1}{2}
\]
$\int^{-1} B = 0 \| -1 \| 0 \| -2 \| -2 \| -2 \| -1$

$B_{1+\frac{1}{4}} = -\frac{1}{2} \配套 - \frac{13}{8}$

$B_{1+\frac{1}{4}} = -1 \| -1\frac{1}{8}$

$B_{1+\frac{13}{16}} = -1\frac{1}{16}; \mu(B) = -1\frac{1}{16}, t(B) = 1\frac{13}{16}$

5C. $\mu(C) = 7\mu(A) + 23\mu(B) = 24\frac{1}{2} - 24\frac{7}{16} = +\frac{1}{16}$.

(What about the temperature?)

5D. $1\frac{7}{8} = \frac{1}{16} + 1\frac{13}{16} \geq \text{Leftstop}(C) \geq \frac{1}{16} \geq \text{Rightstop}(C) \geq \frac{1}{16} - 1\frac{13}{16} = -1\frac{3}{4}$.

But the stops of $C$ are integers, so Leftstop($C$) = 1 and Rightstop($C$) = 0 or -1.

If Right plays first, and plays on a $B$ as long as possible, after 23 moves (12 by Right and 11 by Left) we will have

$$\mu\left(C^{(RL)^{12}R}\right) \leq \mu(C) - t(B) = \frac{1}{16} - 1\frac{13}{16} = -1\frac{3}{4}$$

But when all $B$’s are gone,

$$t\left(C^{(RL)^{12}R}\right) \leq \max\left\{1\frac{1}{4}, 1\frac{1}{2}, 1\frac{7}{8}\right\} = 1\frac{1}{2}.$$  

So then mean + temp $< -\frac{1}{4}$, and

Rightstop($C$) = -1.
Given Thermograph, Find Game

Find a 4-stop optionless game which has the following thermograph

Solution

For the game $G^R$, let us pick $G^R = 0 \mid -2 \mid -5$. Hence we may take

$$G = 3 \mid 0 \mid -2 \mid -5.$$
18 Miscellaneous Notes

Notes on Sums of Games with Mean Value 0

There is some tendency for $2 \cdot g, 4 \cdot g, 8 \cdot g, \ldots$ to be closer to the mean value than $g$. In fact, it is not hard to show that if $g$ has monotonic temperatures (meaning that every ancestor of every position in $g$ has a lower temperature than its descendants), then $2^k \cdot g = 0$ for sufficiently large $k$. But more than this seems to be true. Perhaps one can say something about games with semi-monotonic temperatures (in which for each position $h$, either all $h^L$ or all $h^R$ have lower temperatures). There is empirical evidence that in such games, the difference between the $L(g)$ and $R(g)$ is not greater than $t$, and further indications that for sufficiently large $k$, $2^k \cdot g$ has either $L(g) = 0$ or $R(g) = 0$.

It is common for $2 \cdot g$ to be less confused than $g$. However, one can construct the following game,

$$g = \{51 \pm 50 | \{0, \pm 50\} - 200\}.$$ 

Here the mean is 0 and the temperature is 51. Yet $L(n \cdot g) = n$ for $n = 1, 2, \ldots, 51$ while $R(n \cdot g) = 0$. Thus, for $n < t$, $n \cdot g$ is $n$ times as confused as $g$.

One can also construct games for which the confusion interval of $n \cdot g$ runs from 1 to -1 for all $n$. For example:

$$g = \{2 \pm 1 | \{0\} - 5\}, \{\pm 1\} \pm 5\}.$$ 

But both of the previous examples have temperatures $\geq L(g) - R(g)$, and both use commas. Perhaps these features are necessary for games with semi-monotonic temperatures. If one drops the demand for semi-monotonicity, then we have the following game with temperature 1 and confusion of $n \cdot g$ running from 1 to -1 for all $n$:

$$g = 10^2 || 1 || 0-5 || 9-1 || -10^3$$

**Definition**: A sum of games $\sum_i G_i$ is **irreducible** if no subset sums to zero.

**Theorem**: Let $G + \sum_i \{0|H_i\}$ be an irreducible sum. Suppose $G$ is nonempty and no $H_i \leq 0$. Then if Left has a winning first move on any $\{0|H_i\}$, she has a winning first move on $G$.

**Proof.** If Left wins by playing from $S_0 = \sum_i \{0|H_i\}$ to $S_1 = \sum_{i \neq j} \{0|H_i\}$, then $G + S_1 \geq 0$. But $G + S_1 \neq 0$ since the sum is irreducible, so it follows that $G + S_1 > 0$ and Left has a winning move from $G + S_1$. If it is to $G + S_2 \geq 0$, then we similarly conclude that $G + S_2 > 0$ etc.

So for some $n$, Left must have a winning move from $G + S_n$ to $G^L + S_n \geq 0$. And since $S_0 - S_n > 0$ we have $G^L + S_0 > 0$ so Left could also win by playing from $G + S_0$ to $G^L + S_0$.

**Number Avoidance Theorem** skipped since it’s too common.

"Nim, Remoteness and Suspense in Hot Games" skipped - I believe it can be found in Winning Ways.
Problem. Consider these three games:

\[
A = \pm 101 \pm 100 \pm 45 \\
B = 151|1 || 0|300 \\
C = 200|0 || 50, 40|80
\]

You are to play Right on \( G = i \cdot A + j \cdot B + k \cdot C \), where \( i, j, k \) are positive integers.

1. Graph \( i \cdot L_s(A) + j \cdot L_s(B) + k \cdot L_s(C) \).
2. Graph \( W_x = \max \{W_x(A), W_x(B), W_x(C)\} \).
3. Determine Right’s ambient temperature and his thermostat move on \( G \) (as a function of \( i \) and \( j \) for \( 1 \leq i, j \leq 5 \)).

Solution. The following shows the ambient temperature as a function of \( i \) and \( j \):

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
i & 1 & 2 & 3 & 4 & 5 \\
\hline
i & 1 & 20 & 45 & 45 & 45 & 101 \\
& 2 & 20 & 45 & 45 & 101 & 101 \\
& 3 & 20 & 45 & 101 & 101 & 101 \\
& 4 & 20 & 76 & 101 & 101 & 101 \\
& 5 & 76 & 76 & 101 & 101 & 101 \\
\hline
\end{array}
\]
Cooling Exercises

1. Find the thermograph, temperature, and mean values of each of the following games:

   \[ A = \pm 2 \]
   \[ B = \pm 10 \pm 9 \pm 8 \pm 7 \]
   \[ C = 3 \parallel -2,0\parallel -8 \]
   \[ D = 12\parallel 1 \parallel -2,0\parallel -8 \]

   \[ E = G \parallel -G, \text{ where } G = \begin{cases} 
   2000 \parallel 90 \parallel 2 \parallel 0 \parallel -5 \parallel -1 \parallel -100 
   \end{cases} \]

2. Find Left’s and Right’s ambient temperatures on each of these sums:

   \[ A + B + C + D, \quad B + C + D + E \]

3. How does ambient temperature change, on \( B + C + D + E \), if opponent plays on \( E \)?

Solutions. The following are thermographs of \( A \) to \( D \) (I can’t find the one for \( E \)):
Math 195 Quiz, 1982

The thermograph for $2||0|−10$ is shown below:

1a. Plot the thermograph for $±5 ± 4 = 9|1|−1|−9$.
1b. Plot the thermograph for $8, \{12|6\} \parallel −1, \{3|−10\}$.

Solutions

\[ g = 9 \parallel 1 \parallel −1 \parallel −9 \]
\[ g_t = 9 − 2t \parallel 1 \parallel −1 \parallel −9 + 2t \]
\[ g_4 = 1* \parallel −1* \]
\[ g_{4+t} = 1 − t \parallel −1 + t \]
\[ g_5 = * \]
\[ g = 8, 12|6 \upharpoonright -1, 3|-10 \]
\[ g_t = 8-t, 12-2t|6 \upharpoonright -1+t, 3|-10+2t \]
\[ g_2 = 6, 8|6 \upharpoonright 1, 3|-6 \]
\[ g_{2+t} = 8-2t|6 \upharpoonright 1+t, 3|-6+2t \]
\[ g_3 = 6* \upharpoonright 2, 3|-4 \]
\[ g_{3+t} = 6-t \upharpoonright 2+t, 3|-4+2t \]
\[ g_4 = 5 \upharpoonright 3, 3|-2 \]
\[ g_{4+t} = 5-t \upharpoonright 3+t, 3|2 \]
\[ g_6 = 3|3|2 = 3-1 \]
Math 195 Quiz, 1979

Determine the mean value, temperature and thermograph of

\[ G = \{ \{2|0\} \mid \{0|1\} \mid -4 \} \]

**Solution.** The thermograph for \( G^R \) is

\[ \text{thermograph of } G = 2 \quad 1 \quad 0 \quad -1 \quad G^L \quad G \quad G^R \]

Hence the thermograph of \( G \) is

\[ \text{thermograph of } G = 2 \quad 1 \quad 0 \quad -1 \quad G^L \quad G \quad G^R \]
Math 195 Quiz, 1986

Draw the thermographs for each of the following games:
1a. $3||0|{-10}$
1b. $\{3|-1,\{0|-10\}\}$
1c. $\frac{1}{2} \| \| (\frac{1}{4} + *) \| | {-\frac{1}{4}}$

Solutions.

1a.

1b.

1c.

We have $G = \int \int \frac{5}{8}$, and

$$G_t = \frac{1}{2} - 2t \| \| \frac{1}{4} - t \| t \| - \frac{1}{4} + 2t.$$

$$G_{\frac{1}{4}} = \frac{1}{4} * \| \| \frac{1}{8} * | 0$$

$$G_{\frac{1}{8} + \frac{1}{16}} = \frac{3}{16} | \frac{1}{8} *$$

$$G_{\frac{1}{8} + \frac{1}{16} + \frac{1}{32}} = \frac{5}{32} *$$
Problem. Draw the thermographs of the following games. Show the coordinates of each phase transition.

\[ G = 1\underline{0} -2 \]

\[ H = \begin{cases} 
16 & \{2 \mid -2, \{-1 \mid -11\}\} \\
-5
\end{cases} \]

Solution. For \( H \), we have

\[ H = \begin{cases} 
16 & \{2 \mid -2, \{-1 \mid -11\}\} \\
-5
\end{cases} \]

\[ H_t = \begin{cases} 
16 - 2t & \{2 - t \mid -2 + t, \{-1 \mid -11 + 2t\}\} \\
-5 + t
\end{cases} \]

\[ H_1 = \begin{cases} 
14 & \{1 \mid -1, \{-1 \mid -9\}\} \\
-4
\end{cases} \]

\[ H_{1+t} = \begin{cases} 
14 - 2t & \{1 - t \mid -1 \mid -9 + 2t\} \\
-4 + t
\end{cases} \]

\[ H_3 = \begin{cases} 
10 & \{-1 \mid -5\} \\
-2
\end{cases} \]

But the game \(-1\underline{-1} -5\) is infinitesimally close to \(-1\), so \( H_3 \approx \{10\mid -1\mid -2\}. \) And

\[ H_4 = 8\mid -1 \mid -1. \]

So \( H \) freezes to \(-1\) at temperature 4. The thermograph of \( H \) is:

![Thermograph of H](image-url)

68
Subtle Facts About Temperature Theory

1. The opponent may be able to force large increases in the temperature of many components of an apparently cool sum. E.g.

\[ G = 5 || \pm 100 \pm 99 | -151. \]

![Diagram showing temperatures and moves](image)

The temperature of \( G \) is 6, but Right can heat it up to 100. Even if \( G \) is added to

\[ H = 5 || \pm 100 \pm 99 | -1000, \quad \text{and} \quad K = 5 || \pm 100 \pm 99 | -2000, \]

Right can heat up ALL of \( G, H, K \) in the sum \( G + H + K \).

2. Even when the temperatures of all summands are strictly decreasing from each position to its followers, the best move may be on the coolest summand (see page 167 ???). Note the bug in the last sentence of the first ?? on page 169 of Winning Ways (edition 1). The phrase “move gives Right -89 or better” should be “move gives Left at most 13”. (Is 13 correct?)