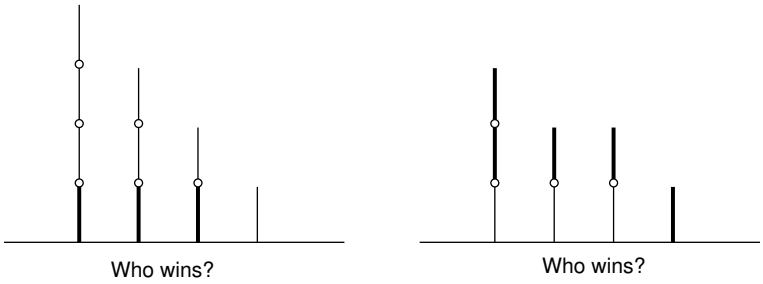
Hackenbush

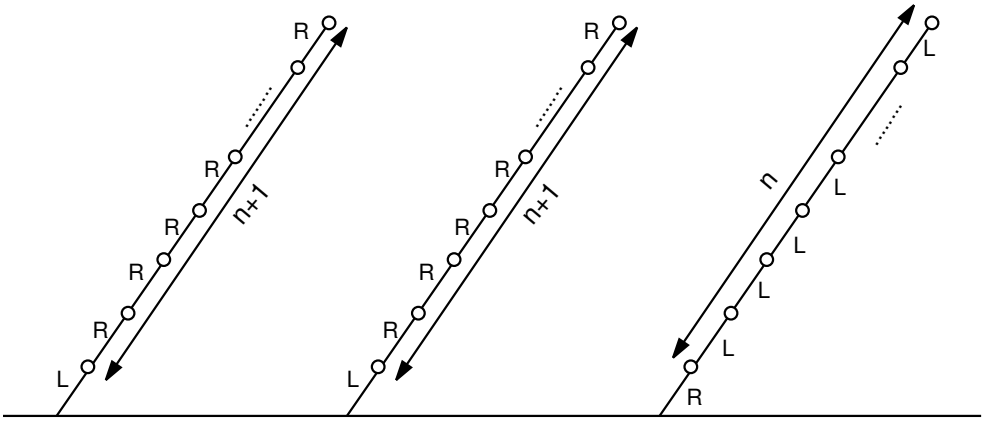
1 Warm-ups

Here are 2 Hackenbush games. For each one decide whether Light or Dark ought to win if the game is played well. Then write your conclusion in the blank by the game. Be prepared to defend your decision by playing the game.



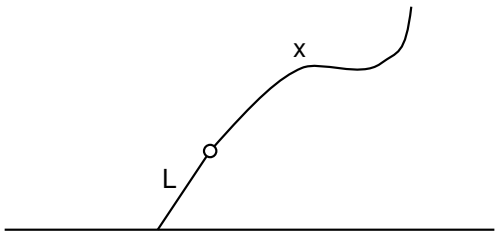
A Basic Problem

Prove that if a finite Red-Blue Hackenbush string contains several Red branches, then Red's best move is the branch furthest from the ground. By induction on n, prove that the second player can win from this:



2 Grownup Left-Right Hackenbush

Let x be represented as a Hackenbush string. Plot the function $f(x)$ which has the same value as this Left-Right Hackenbush graph:



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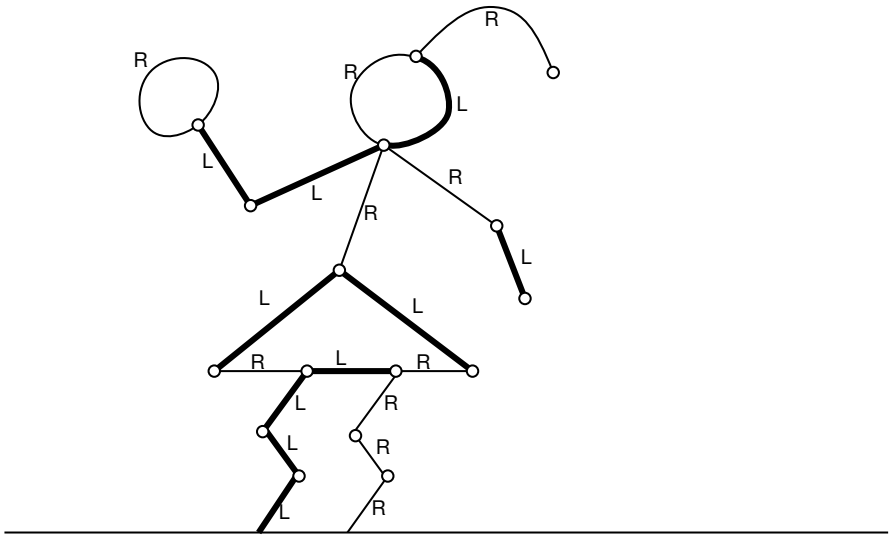
Solution.

x		Lx	
positive			$1 + f(x)$
0		L	1
$-\frac{1}{2}$	RL	LRL	$\frac{3}{4}$
-1	R	LR	$\frac{1}{2}$
$-1\frac{1}{2}$	RRL	LRRL	$\frac{3}{8}$
-2	RR	LRR	$\frac{1}{4}$

The function is *piecewise linear*. The derivative $\frac{d(Lx)}{dx}$ is discontinuous at $x =$ negative integer.

Left-Right Hackenbush Problem

Determine the value of the following grownup Left-Right Hackenbush position, which is copied from the first page of Chapter 1 of the book Winning Ways:



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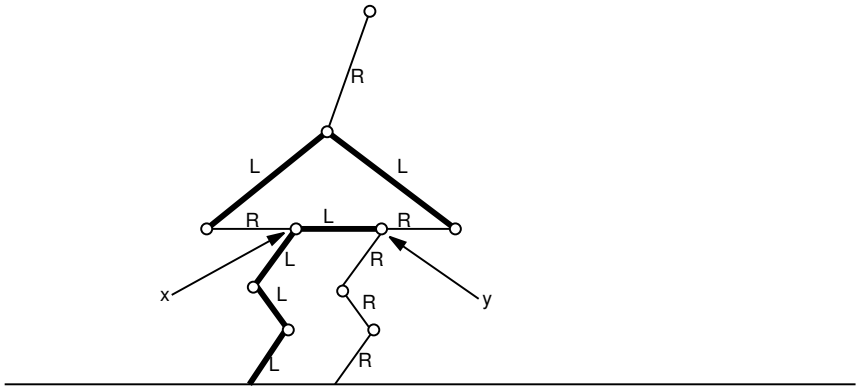
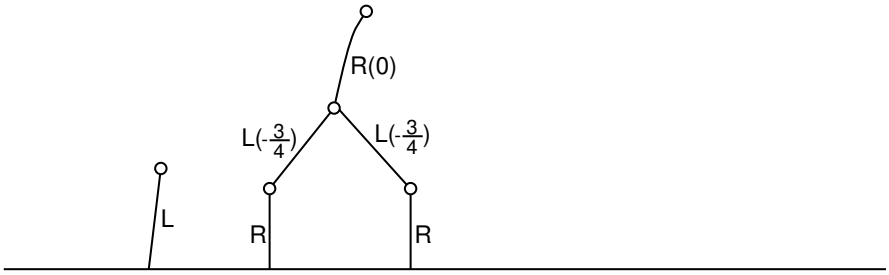
Solution. We have:

$$\begin{aligned} \text{Right arm and racket} &= 1\tfrac{1}{2} \\ \text{Head and pony-tail} &= -1 \\ \text{Left arm} &= -\tfrac{1}{2} \end{aligned}$$

which sums up to zero, so we may sever everything above her neck.



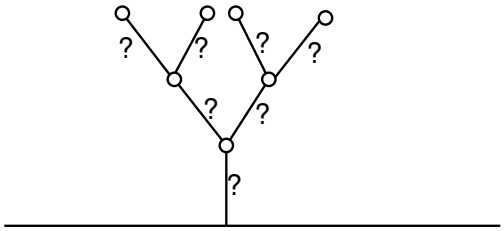
But, since the graph supported at x is positive, we may merge x and ground. Then the Blue branch at y may be disconnected from y, and y also fuses to the ground. This gives:



The values of the followers are shown above. Evidently, the girl has value $-\frac{3}{4}|0 = -\frac{1}{2}$.

Labelling Problem

This Blue-Red Hackenbush tree has 4 Blue branches and 3 Red branches. When converted to canonical form, its birthday is 7.
a. Label the tree's branches.
b. What is the equivalent L-R Hackenbush string?



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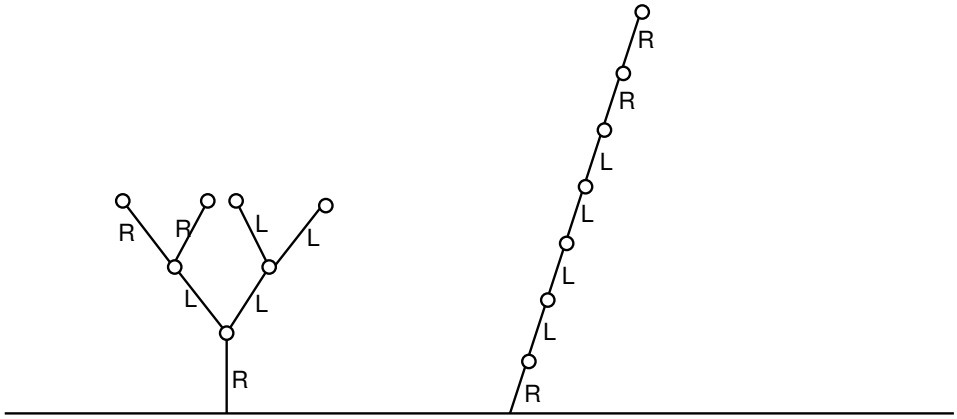
Solutions. This Blue-Red Hackenbush tree has 4 Blue branches and 3 Red branches. When converted to canonical form, its birthday is 7. The equivalent L-R Hackenbush string is shown on the right.



The crux is the observation that, if x and y are numbers represented as Hackenbush strings, then

$$Birthday(x + y) \leq Birthday(x) + Birthday(y)$$

and that equality occurs ONLY IF x and y have the same sign AND at least one of them is an integer.

So, in the tree shown, the two branches emanating upward from any node must have same color. Therefore the trunk's color must occur an *odd* number of times. If the other two red branches touch the trunk, the numbers touching the trunk are nonintegers. So they must be and the solution is essentially unique.

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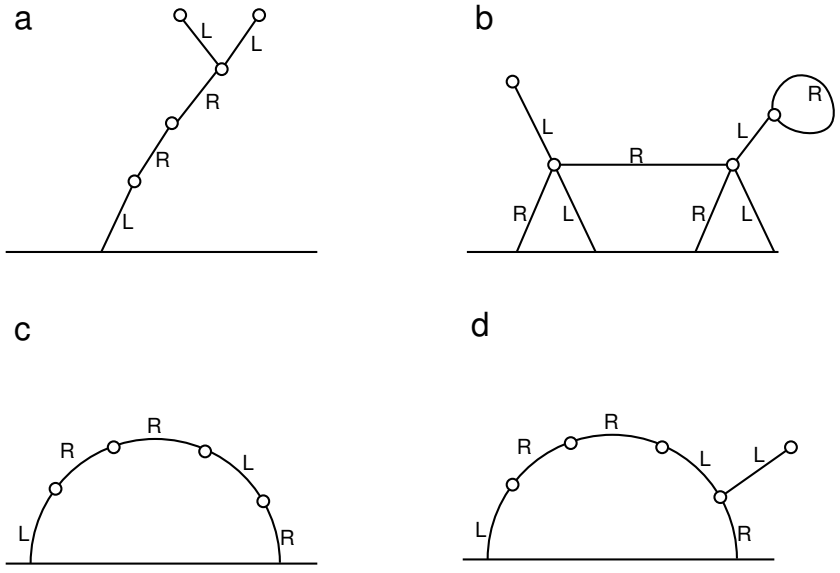
For each of the following positions in Left-Right Hackenbush, determine the equivalent rational number.



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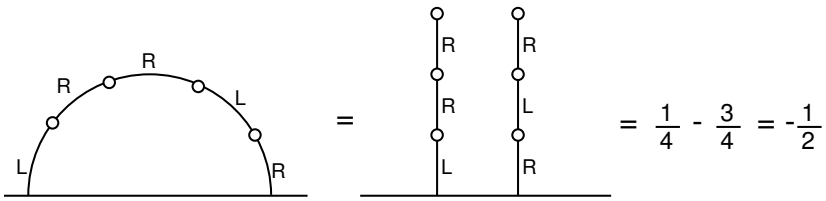
Solutions. a. LRRLL $= +.0111 = \frac{7}{16}$

b. Since all other branches are grounded through their own colors, Right's best move is the dog's head, and Left's best move is the neck. In either case, the resulting game is easily seen to be an integer. Thus $G = \{0|1\} = \frac{1}{2}$.

c.

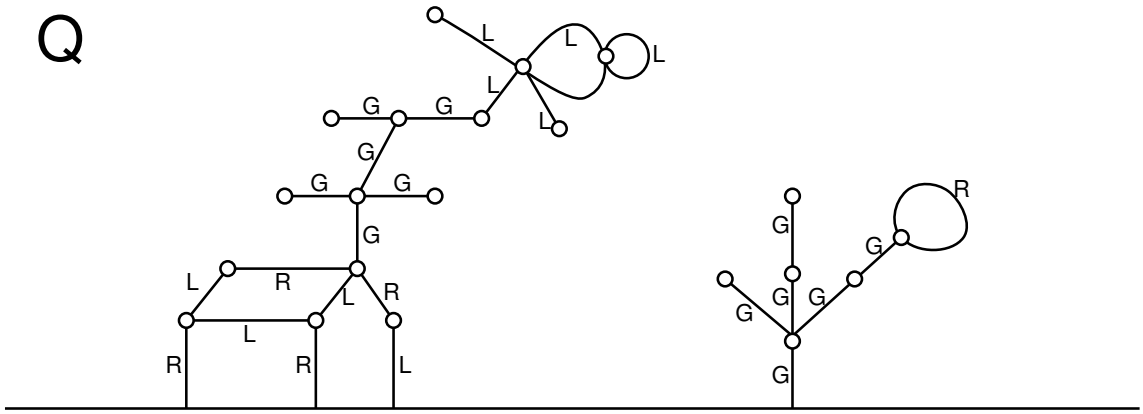


or $G = \{-\frac{3}{4}|0\} = -\frac{1}{2}$.

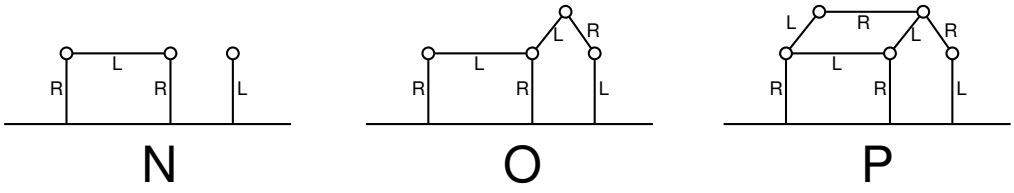
d. Now, $G = \{-\frac{1}{4}|\frac{1}{4}\} = 0$.

Introductory Hackenbush Diagram

The following picture shows a Hackenbush Hotchpotch position which Berlekamp often uses in his first demonstration in introductory lectures of game theory:



If all green branches and everything supporting them are removed, only a BLue-Red House remains. Find the values of these positions of the house. [Detailed argument unnecessary, right answers for parts receive full credit.]



Q. In the full starting position shown above, would you choose to play Left, Right, 1st, 2nd?

R. Explain your plans for your first move(s).

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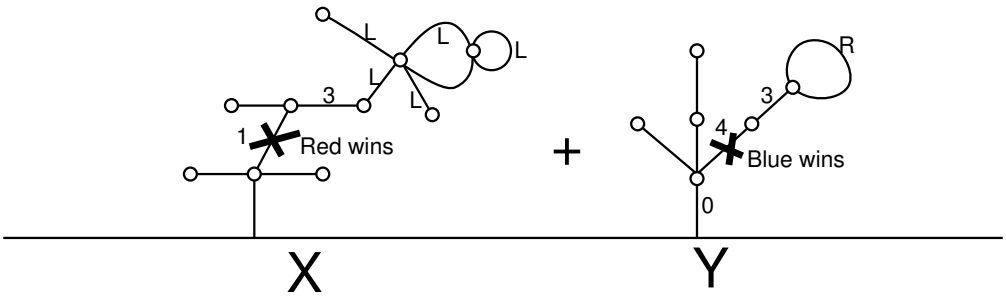
Solutions. $N = 0, O = 0, P = 0$.
The Green jungle (aerial and Bluebird) slides down the Purple mountain (house) to give



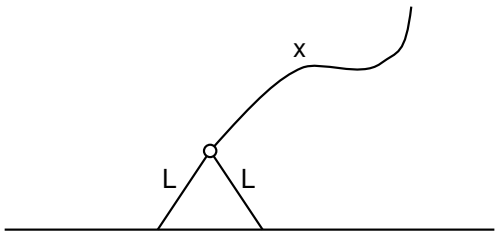
where all unlabelled branches are green. All moves on green which change atomic weights are labelled with the size of equivalent nimheap which that picture attains after such a move is made.

Right can attack X, leaving $*0$, $*1$ or $*3$.

Blue (Left) can attack Y, leaving $*0$, $*4$ or $*3$.

Q-R. Either player going *FIRST* can win by converting his opponent's-colored "flower" into a nim-heap which his opponent cannot match. Red first can win only by cutting the *middle* of the aerial; Blue first can win only by cutting the *middle* of the tree.

Problem. Let x be represented as a Hackenbush string. Plot the function $f(x)$ which has the same value as this Left-Right Hackenbush graph:



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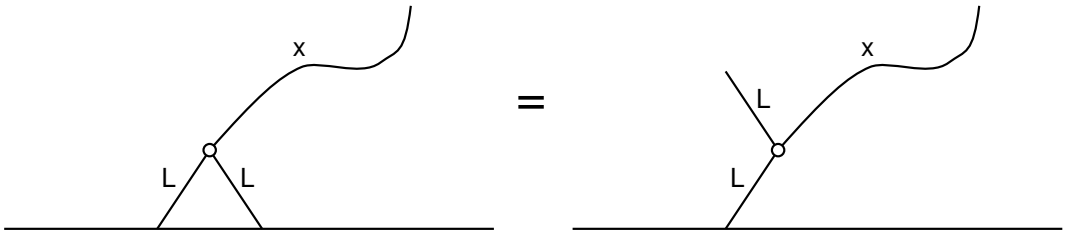
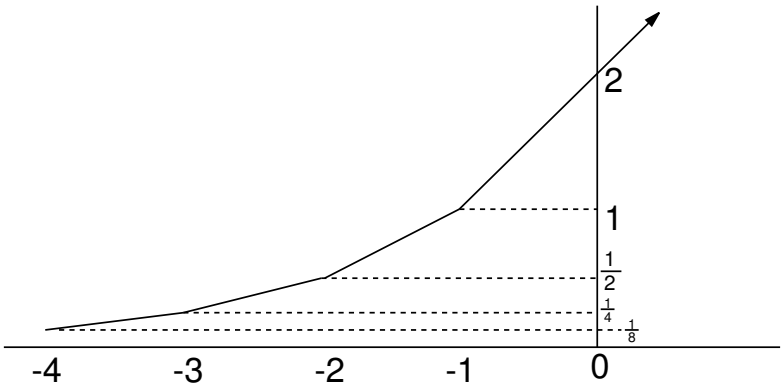
Solution. Note that:

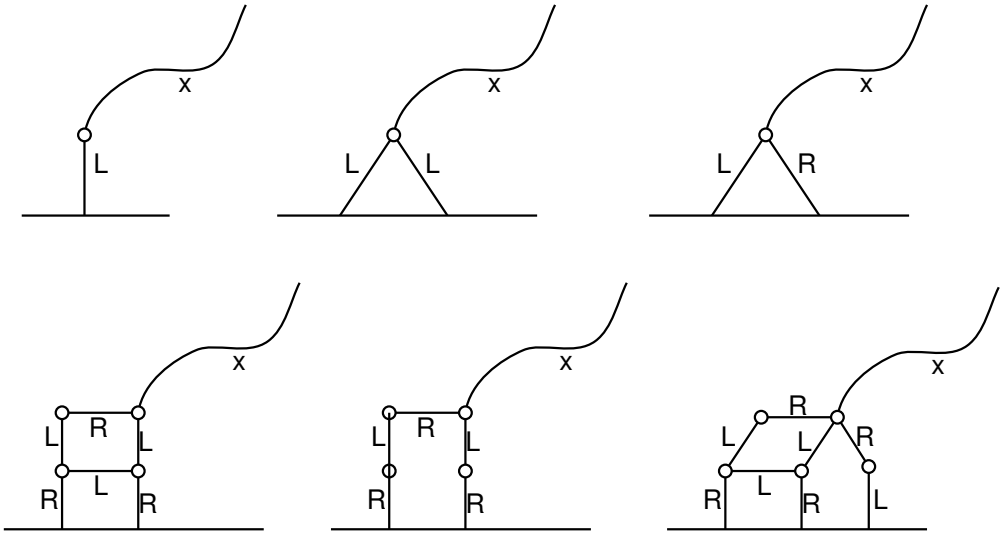


We get the following graph:



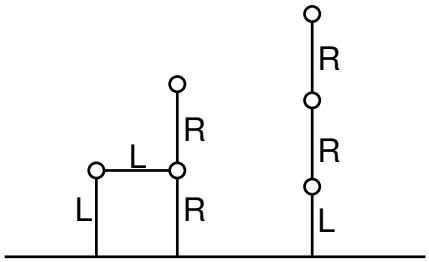
Hackenbush Graphs

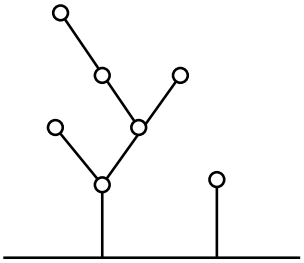
Let x be an L-R Hackenbush string whose value is equal to the real number x'. Plot the function of x' corresponding to each of the games shown below.



Hackenbush Hotchpotch Problems

You are to play grown-up Hackenbush Hotchpotch. Any branch not labelled L or R is Green. You are Left. It is your turn. In each of the following grown-up Hackenbush Hotchpotch positions, mark a winning opening move if you can find one; otherwise "RESIGN".





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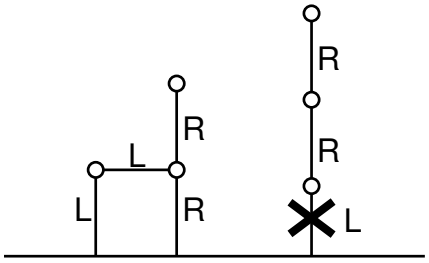
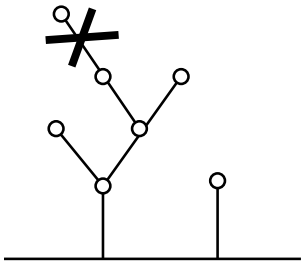
Solutions



Then the chair $= 0$.



Then $* \ + \ * = 0$.