1 Impartial Games

1.1 Impartial Cleric Problem

A cleric\(^1\) is an impartial piece which can be moved only one square in any of three directions: N, W or NW. Several clerics may occupy the same square. The game ends when all clerics reach the [0,0] square, and the player unable to move loses.

Problems:

1. Find a formula(s), in terms of \(n\) and \(m\), for the nim-heap equivalent to a cleric located at position \([n, m]\).

2. Find ALL winning moves from the above starting position with four clerics shown.

3. Suppose there is only one cleric at (75,70). What move do you make from this position?

4. Suppose there is only one cleric at (75,85) and the game is played in conjunction with a remote star \(\star\). What move do you make from this sum?

\[\text{Solution on following page}\]

\[^1\text{A partisan cleric is called a duke}\]
Solutions: (using the “mex” rule)

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The pattern of 1’s and 3’s is seen to switch depending on which side of the main diagonal it is on.

\[ c(\text{even, even}) = 0, \]
\[ c(\text{odd, odd}) = 2, \]
\[ c(\text{odd, even}) = c(\text{even, odd}) = 1, \text{ if odd > even} , \]
\[ c(\text{odd, even}) = c(\text{even, odd}) = 3, \text{ if even > odd} . \]

2. The relevant starting positions are circled and the incentives are shown on the right:

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| 4 | 0 | 3 | 0 | 3 | 0

\[ \begin{array}{c}
\text{WINNING move and it is unique} \\
\end{array} \]
```

\[ \text{Starting value} = 1 + 2 + 2 + 0 = 1 \]
\[ \text{Winning value} = 0 \]
\[ \text{Needed incentive} = 1 \]

So the ONLY winning move is \([0,3]\) to \([0,2]\).

3. \((75,70) = (5,0) = *\)
   Move to \((74,70) = 0\)

4. \((75,85) = (1,15) = *2\)
   Move \(\star\) to \(*2\)

Our 2-dimensional impartial cleric is closely related to the game of “Princess and Roses” described in volume 2 of Winning Ways First Edition. Results mentioned there offer hope that clerics might be generalized up to 5 dimensions in a way that continues to yield very tractible solutions.
1.2 Squires

A squire is a chessman who, not yet having been promoted to knighthood, can move from coordinates \((x, y)\) only to \((x + 1, y - 2)\) or to \((x - 2, y - 1)\) or to \((x - 2, y + 1)\). Two players alternately move an impartial squire in the first quadrant of the Diophantine \((x, y)\) plane until it reaches \((0, 0)\) or \((0, 1)\) or \((1, 0)\) or \((1, 1)\), when the player unable to move loses.

Problems:

1. The squire is at \((6, 6)\) and it is your turn. Where do you move?

2. The squire is at \((10, 10)\) and it is your turn. Where do you move?

3. Determine the set of starting positions of the squire from which the second player wins the sum of the squire game and $5$.

??????????????? Solution on following page ?????????????????
Solutions:

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If \( x \geq 4 \) and \( y \geq 4 \), then \( G(x, y) = G(x - 4, y - 4) \).
If \( x > 8 \) and \( y < 4 \), then \( G(x, y) = G(x - 4, y) \).

1. From \((6,6)\), go to \((5,4)\) or \((4,5)\).
2. From \((10,10)\), go to \((9,8)\) or \((8,9)\).
3. None. Since a squire has only 4 moves, no position can have nimber more than 4. (As shown above, the only positions of nimber 4 are \((4k, 4k - 1)\) and \((4k - 1, 4k)\)).
1.3 Odious Nim Problem

Recall that an integer which has an odd number of ones in its binary expansion is called \textit{odious}; an integer which has an even number of ones in its binary expansion is called \textit{evil}. The odious numbers in the sequence along the table at the bottom are marked with an 'X'.

\textbf{Odious-Nim} is an impartial game played on several heaps of counters. Each legal move consists of removing any odious number of counters from a single heap.

\textbf{Problem:} Using Grundy scales (or any other method you prefer) compute sufficiently many nimbers for this game to determine ALL winning moves, if any, from each of the following positions:

1. There are 3 odious-nim piles: 1, 2 and 3.
2. There are 3 odious-nim piles: 11, 12 and 13.
3. There are 3 odious-nim piles: 21, 22 and 23.

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
X & X & & & X & X & & & X & & & X & & \\
\hline
\end{tabular}
\end{center}

Grundy scales (the odious numbers are marked with ‘X’)

??????????????? Solution on following page ?????
Solution: After extending the sequence of odious numbers in the problem, we may slide that scale along the one below, and employ the “Grundy scale” method to compute the nimbers shown along the table below.

\[
\begin{array}{ccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 3 & 4 & 5 & 3 & 4 & 5 \\
\end{array}
\]

Although many nim sequences become permanently periodic after an initial transient, this one gets interesting only after an initial (fickle) exhibition of seven complete cycles of apparent periodicity!

This unusual behavior arises from these facts:

1 and 2 are odious, but 21 is the first odious multiple of 3.

From the nimbers, it is straightforward to find the winning moves:

1. 2 → 1 (unique).

2. 11 → 10, 7 or 4 by taking 1, 4 or 7, OR
   13 → 11, 7, 5 or 2 by taking 2, 8 or 11.

3. 21 → 19, 13, 10 or 7 by taking 2, 8, 11 or 14 leaving a \(p\)-position with these nimbers:

   odious[23] = 11 (3 in binary)
   odious[22] = 10
   odious[21] = 01

which has a Nim sum of zero.
1.4 Socialistica

In the two-pile impartial game of Socialistica, the following are the legal types of moves:

(a) take away any number from the first pile,
(b) take away any number from the second pile,
(c) take away the same number from both piles,
(d) take any number from the second pile to the first pile.

If there are $x$ counters in the first pile and $y$ in the second, then these moves replace $x, y$ by

(a) $(x - i, y)$, where $0 < i \leq x$,
(b) $(x, y - i)$, where $0 < i \leq y$,
(c) $(x - i, y - i)$, where $0 < i \leq \min(x, y)$,
(d) $(x + i, y - i)$, where $0 < i \leq y$.

**Problem:** Determine the nimber value of the position with $x = y = 2$.  

????????????????? Solution on following page ???? ????????????????
Solution:

\[
\begin{array}{ccc}
1 & 4 & 5 \\
2 & 3 & 0 & 1 \\
0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

So at \( x = y = 2 \), the nimber is 5.
1.5 DeBruijn’s Kalah-Nim

This game is played with coins on a discrete line, at the end of which is a money bag. Several pennies and a silver dollar are initially placed at various positions along the line. A legal move for either player is to move any single coin toward the money bag, but the moved coin may not land on top of or pass over any other coin(s) still on the line. A coin moved into the bag becomes the property of the player who moved it there, and the object of the game is to capture the silver dollar.

**Problem:** Determine and describe an optimum strategy for playing Kalah-Nim.

???????????????? Solution on following page ?????????????????

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Solution: Refer to the Kalah-Nim diagram below, where the money-bag is on the extreme left.

Start furthest from the money-bag, and measure alternate intervals. If there is an odd number of coins, the last interval stops just short of the money-bag (if the dollar-coin is second from left) or includes the money-bag otherwise.

Play Nim on the sizes of these intervals. The reason that the measurement of the last interval is irregular when there is an odd number of coins is that taking the penny in front of the dollar-coin is a disastrous move!
1.6 Conway’s Whim

In this game there are several heaps of counters and a blank card. At each turn, either player may remove counters from any single one of the heaps, or he may instead write either of two things on the card: “Last counter wins” or “Last counter loses”. Once the card is written on, it cannot be written on any more. The game ends when the card is written and all counters are gone, and the winner is determined by the writing on the card.

**Problem:** Determine an optimum strategy for playing Whim, and explain it.

????????????????? Solution on following page ???????????????????
Solution: Heuristically, the blank card behaves rather like a Nim-heap, but its value depends on its environment.

When there are many large piles about, either written card is essentially the same by the analysis for Misere Nim. Hence, in an environment containing sufficiently many large piles, the blank card has value *1.

But, when all conventional piles have 0 or 1 counters, the written card may have value 0 (if it says last counter wins) or value *1 (if it says last counter loses). Thus, in this environment, the blank card has value *2.

Thus, the effective value of the blank card is *1 at the beginning of a typical game with several large heaps, but it may increase to value *2 at most once in the course of play. When? To decide, play the blank card against one or two heaps of sizes 2, 3 or 4.

If there are ≥ 2 heaps containing 4 or more counters, treat the blank card as *1.

If all heaps contain ≤ 3 counters, treat blank card as *2. Don’t move on the card if there is another winning move. If the opponent moves from the card of value *2, he leaves an odd number of heaps containing 2 or 3, and we win.
1.7 Dawson-Like Reflexions

Here is the sequence of nimbers which solves Dawson’s game:

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This sequence has the property that for certain values of the point \( p \), including \( p = 5, 15 \) and 32, one observes the reflexion phenomena:

\[ D_n = D_{2p-n} \]

for many (but sometimes not quite all) of the values \( n = p + 2, p + 3, \ldots, 2p - 1 \).

We now seek to “explain” this phenomena, by showing that if the above equation holds for several values of \( n \), then there is a strong tendency for this to continue. More specifically, although the hypotheses of the following theorem allow one only to conclude that \( D_n \leq D_{2p-n} \), the large number of pseudo-randomly distributed small values in the “mex” which defines \( D_n \) suggests that if the (small) value \( D_{2p-n} \) is not excluded, it is likely to prevail.

**Problem:** Prove the following “Reflexion Theorem”.

**Definition:** Consider the game in which either player can remove \( s \) counters from the middle or end of a single pile. Then a pile of length \( n \) has nim-value

\[ D_n = \text{mex}_k \left( D_k^* + D_{n-s-k} \right). \]

**Reflexion Theorem:** For any integer \( p \), and for any integer \( n \) such that \( p+s < n \leq 2p+1-s \), one of the following holds:

(i) \( D_n \leq D_{2p-n} \).

(ii) There exists \( k, 2p - n < k \leq n - s \) (which implies \(|k - p| < |n - p|\)) such that

\[ D_k \neq D_{2p-k} \quad \text{and} \quad D_k^* + D_{2p-n}^* + D_{n-s-k} = 0, \]

(iii) There exists \( k, \frac{n-s}{2} < k < 2p - n \) (which implies \( n < \frac{4p+s}{3} \)) such that

\[ D_k^* + D_{2p-n}^* + D_{n-s-k} = 0. \]

**Note:** In several applications of interest, the appearance of an exceptional \( k \) (such that \( D_k \neq D_{2p-k} \)) may fail to proliferate further exceptions to the reflexional pattern because the values of \( D_k \) and \( D_{2p-k} \) are so large relative to other relevant \( D \)'s that for most \( n \)'s we are assured that

\[ D_k > D_{2p-n}^* + D_{n-s-k}. \]
In fact, the appearance of \( s - 1 \) big values at \( D_{p-s+1}, D_{p-s+2}, \ldots, D_{p-1} \) and another \( s - 1 \) big values at \( D_{p+1}, \ldots, D_{p+s-1} \) commonly occurs in conjunction with a reflexion around \( p \).
Solution: If $D_n > x$, then the definition of $D_n$ ensures that there exists a $k$ such that 
$x = D_k + D_{n-s-k}$ and without loss of generality, we may assume that $k \geq n - s - k$. In particular, if $D_n > D_{2p-n}$ then there exists $k$, such that $\frac{n-s}{2} \leq k \leq n - s$, and

$$D_{2p-n} + D_k + D_{n-s-k} = 0.$$ 

We may partition the claim into two cases:

Case 1: $2p - n < k \leq n - s$.
Case 2: $\frac{n-s}{2} \leq k \leq 2p - n$.

In case 2, we show that the bounds on $k$ may be replaced by strict inequalities. For if $k = \frac{n-s}{2}$, then $D_k = D_{n-s-k}$ and we have

$$D_{2p-n} = 0, \quad \text{and } 2p - n \equiv 0 \pmod{2}.$$ 

Similarly, if $k = 2p - n$, then

$$D_{n-s-k} = 0, \quad \text{and } n - s - k \equiv 0 \pmod{2}.$$ 

Both of these subcases contradict the fact that if $j \equiv s \pmod{2}$, and if $j \geq s$, $D_j \neq 0$ because there is a move to

$$D_{(j-s)/2} + D_{(j-s)/2} = 0.$$ 

In case 1, we have

$$2p - n < k \leq n - s, \quad \text{and } D_{2p-n} + D_k + D_{n-s-k} = 0.$$ 

Observe that the definition of

$$D_{2p-k} = \text{mex} \left( D_i + D_{2p-k-i-s} \right)$$

ensures if we set $i = 2p - n$, that

$$D_{2p-n} + D_{2p-k} + D_{n-s-k} \neq 0,$$

where $D_{2p-k} \neq D_k = D_{2p-n} + D_{n-s-k}$.

Note: with $s = 3$, reflexion-like tendencies appear at $p = 8$ and at $p = 24$. 

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1.8 Three-Pile Nim Ending With Two Sticks

In regular Nim, each player takes as many sticks as he wishes from any single pile, and the game continues until all sticks are gone. The player who is then unable to move loses.

Misere Nim may be considered as the same as Nim, except that the game ends when there is a single stick left. The next player to move loses.

An obvious generalization is Nim til $k$ sticks are left. When there are exactly $k$ sticks remaining (in any number of piles), the game ends, and the next player to move loses.

We consider only the three-pile variation with $k = 3$. Let $x$, $y$ and $z$ be the number of sticks in the three piles. It is easy to see that second player can win if $z = f(x, y)$, where

\[
\begin{align*}
  f(0, 2) &= f(2, 0) = f(1, 1) = 0, \\
  f(1, 0) &= f(0, 1) = 1, \\
  f(0, 0) &= 2,
\end{align*}
\]

and if $x + y \geq 3$, then

\[
f(x, y) = \max \begin{cases} 
  f(x', y), & x' < x \\
  f(x, y'), & y' < y.
\end{cases}
\]

Even though the function $f(x, y)$ is symmetric, it turns out to be more convenient to calculate $h(x, y) = f(x, y) - x$. For fixed $y$, $f(x, y)$ goes to infinity with increasing $x$ but $h(x, y)$ remains bounded. For example, the sequence

\[
h(0, 0), h(1, 0), h(2, 0), h(3, 0), \ldots
\]

is $2, 0, -2, 0, 0, 0, \ldots$, so we say that $h(x, 0) = 2, 0, -2, 0$. The eventual period is 1, and the last exception occurs at $x = 2$. The fact that $f(x, y) = v$ iff $f(v, y) = x$ translates into the fact that $h(x, y) = r > 0$ iff $h(x + r, y) = -r < 0$, and this fact can be used to obtain a faster recursion for computing $h(x, y)$ for modest $y$ and very large $x$. Computation shows that

\[
\begin{align*}
  h(x, 0) &= 2, 0, -2, 0, \\
  h(x, 1) &= 1, -1, 0, 1, -1 \\
  h(x, 2) &= 0, 1, -1, 2, 2, -2, -2 \\
  h(x, 3) &= 3, 3, 3, -3, -3, -3, 1, -1 \\
  h(x, 4) &= 4, 2, 4, -2, -4, 2, -4, -2, 3, 3, 3, -3, -3, -3, 2, 3, -2, 2, -3, -2
\end{align*}
\]

For small $y$, the asymptotic periods of $h(x, y)$ and the values of $x$ corresponding to the last irregularities are as follows:
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<th>$y$</th>
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The computations were not carried far enough to be certain about the next few rows, but there is empirical evidence to suggest that when $y = 20, 21, \text{ and } 22$, the period of $h(x, y)$ is $8 \times 9 \times 10 \times 11$. When $y = 23$, the period appears to be 24, apparently starting much sooner, apparently after a final exception at 221. Periods for $y = 24, 25, 26, \text{ and } 27$ appear to be 24, 26, 364, 364 respectively.

These results make it quite clear that if a game of Nim is declared over when exactly two sticks remain, the game is much more complicated than if it ends when zero or one stick remains.
2 More Cold and Tepid Games

2.1 Amit Sahai’s “Plodding Kings”

Plodding Kings is a very degenerate variation of Amazons. It is played by 2 Chess Kings on an arbitrary field. In each turn, a King may move one square in any of the 8 directions, but the square from which he moved becomes unusable by any player for the remainder of the game. A King may destroy an enemy King if the enemy is in an adjacent square. The last player to move wins.

Problem: What is the value of a $2 \times n$ board with pieces positioned as follows?

```
L . . . . . . . R
```

????????????????? Solution on following page ??????????????????
Solution: For \( n = 1, 2 \) obviously the value is \(*\).

For \( n > 2 \), \( n \) odd, simply note that a tweedledum-tweedledee strategy works, hence the \( 2 \times (2n + 1) \) case has value 0. For \( n \) even, simply observe the following:

1. The tweedledee-tweedledum strategy applies on odd boards, even if L, R are on the same row, so these games = 0.

2. It is impossible to “cross the enemy” since the height of the board is only 2, hence if either player moves horizontally, the column left behind is unusable by the enemy.

3. A vertical move here is bad, since 2 horizontal moves by the enemy will lead to a position that by 2. and 1. and the odd case, loses for the first player (see figure: if L moves first, he loses)

4. A horizontal or diagonal move, by 2., 1., and the odd case, leads to a game of value 0.

Hence, the \( 2 \times 2m \) case has value \(*\).
2.2 Shearer’s Hexapawn

Hexapawn with optional capture (pre Chapter 12, page 376 →)

We define a sequence recursively by

$$a_n = \begin{cases} a_{n-1}, & \text{if } n \equiv 0, 3 \pmod{5} \\ + a_{n-1}, & \text{if } n \equiv 1, 2, 4 \pmod{5} \end{cases}$$

starting from the initial condition $a_0 = 0$. Obviously, $a_n$ has period 5 and saltus *, or period 10.

Claim. If $a_i + a_j + a_{i+j+3} = 0$, then $i$ or $j$ (or both) is congruent to 0 or 3 mod 5.

Proof. We check 6 cases:

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>i+j+3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>11</td>
</tr>
</tbody>
</table>

\[
\begin{array}{cccccccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
 a_n & 0 & * & 0 & 0 & * & * & 0 & * & 0 & * & 0 & * & 0 \\
\end{array}
\]

Definition. With optional capture / first queen on stalemate wins, define $G_{-1} = G_0 = 0$ and

$$G_m = \begin{array}{cccc}
 d & d & d & \ldots & d & d \\
 p & p & p & \ldots & p & p \\
\end{array} \text{ with } n \text{ columns.}$$

$$H_m = \begin{array}{cccc}
 d & d & d & \ldots & d & d \\
 p & p & p & \ldots & p & p \\
\end{array} \text{ with } n+1 \text{ columns altogether.}$$

Theorem. $G_n = a_n$.

Proof. Let the opponent open on $G_n + a_n$. He must try to move to $G_k + G_{n-3-k}$. If $a_k + a_{n-3-k} \neq a_n$, cooperate in completing his move to $G_k + G_{n-3-k}$ by capturing until that position is reached.

If $a_n = a_k + a_{n-3-k}$, then without loss of generality, we may assume that $k \equiv 0, 3 \pmod{5}$. We first capture in a way that creates $G_{n-3-k}$, but then we advance to force him to move from

$$H_k + G_{n-3-k}.$$
He must move on $H_k$, and he has only two choices: to $H_{k-1}$, or to anywhere via $G_{k-1}$. Evidently, since $a_{k-1} = a_k$, it follows that $a_{k-1} + a_{n-3-k} = a_n$, so he cannot cause trouble via $G_{k-1}$. If he moves to $H_{k-1}$, we get the turn and we move from $G_{k-2}$, which is OK because

$$a_{k-2} \neq a_{k-1}$$

where

$$a_{k-2} + a_{n-3-k} \neq a_n.$$ 

\[\square\]

**Note:** He is Left; we are Right:
2.3 Colored Trominos

1. The game of colored dominoes is played on an \( n \times 1 \) board. Each player has a large supply of dominoes of his color. At each turn, he must place a domino onto two adjacent vacant squares in such a way that it does not touch previously placed dominoes of the same color. Solve the game. What happens if you use monomoes instead? What if each player has both pieces?

2. Likewise, colored trominos is a game played on \( n \times 1 \) strip with colored edges. Each player, at his turn, plays a \( 3 \times 1 \) tromino of his own color onto an unoccupied triplet of adjacent squares, subject to the rule that Trominos of the same color may not touch each other.

Formally, \( L0R = 0 \) but \( L0L \) and \( R0R \) are impossible, and

\[
L0L = \{LkL + L(n - k - 3)L \mid LkR + L(n - k - 3)R\}
\]

\[
L0R = \{LkL + L(n - k - 3)R \mid LkR + R(n - k - 3)R\}
\]

\[R0R = -L0L\]

Define the condition \( c[n] \) as

\[
\begin{cases} 
<, & \text{if } L0R < R0R \\
=, & \text{if } L0R = R0R
\end{cases}
\]

The values of \( L0R \) are as follows:

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L0R )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>( c[n] )</td>
<td>&lt;</td>
<td>=</td>
<td>=</td>
<td>&lt;</td>
<td>=</td>
<td>=</td>
<td>=</td>
<td>=</td>
<td>&lt;</td>
</tr>
</tbody>
</table>

Find recursions that express these pair sequences (\( L0R \) and \( c[n] \)) in terms of each other, without reference to values of \( R0R \) or \( L0L \).

????????????????? Solution on following page???????????????????

22
Solution. If \( G < *m \), then \( 2 \cdot G < 0 = (*m) + (*m) \) so if \( G \) is less than any nimber, it must be less than or incomparable to all others.

Assume that for each \( k < m \), \( LkR \) is a nimber and that
\[
LkL \leq LkR \leq RkR.
\]

Then
\[
LnR = * \left( LkL + L(n - 3 - k)R \mid \text{max } k \text{ such that } c[k] \text{ is } "="ight)
\]
because the terms where \( c[k] \) is "<" are gift horses. Under the same hypothesis:
\[
c[n] = \begin{cases} 
  "=" \text{ if } \text{mex}_{k}\{LkR + L(n - 3 - k)R\} = \text{mex}_{k,c[k]=c[n-3-k]="="}\{LkR + L(n - 3 - k)R\} \\
  "<" \text{ otherwise}
\end{cases}
\]

Note: the solution of colored trominoes \( LnR \) is purely periodic, with period 26 (actually period 13 with saltus *).
2.4 Colored Ominoes

The game of colored ominoes is played on $n \times 1$ strips. The rules of the game define two sets. One set is the lengths of Blue and Red ominoes. The other set is the set of Green ominoes. A legal move is to play an omino of your color (or green) onto the board, subject to the condition that you cannot play 2 Red or 2 Blue pieces adjacent to each other.

Positions may be denoted by $LnL$, $RnR$, $LnR$, $LnG$, $RnG$, $GnG$, etc, where $n$ is the number of empty nodes between the specified boundaries.

Open Problem: Pick one or two simple rulesets and calculate all of the values (if you can!)

It may be convenient to assign a number to each such game, $A_1A_2A_3\ldots$, where $A_n$ describes the types of pieces available to each player:

- $A_n = 0$ means there are no $n \times 1$ pieces;
- $A_n = 1$ means each player has $n \times 1$ pieces which have his own color on both endpoints;
- $A_n = 2$ means that each player has $n \times 1$ pieces which have his color on one edge and green on the opposite edge (they may be played in either direction);
- $A_n = 3$ means that each player has $n \times 1$ green pieces.

For example, the game “Seating Couples” is game number .01.

After several moves, the board will decompose into disjoint pieces. Each piece will be one of the following forms:

$$LnL \leq LnG \leq LnR \leq RnG \leq RnR$$

$$LnG \leq GnG \leq RnG$$

Since $LnL = -RnR$ and $LnG = -RnG$, there are only four potentially different sequences: $LnL$, $LnG$, $LnR$ and $GnG$.

Problem: Prove or disprove that $LnR$ and $GnG$ are both nimbers, and that if $LnG$ is a nimber, then $LnG = LnR = GnG$. Furthermore, if $LnL$ is a nimber, then $LnL = LnG$.

Thus, the values of all nimbers arising in this game may be recorded in the form of two sequences of nonnegative integers representing $LnR$ and $GnG$, and two sequences of bits, which specify whether $LnL$ is a nimber and whether $LnG$ is a nimber. For convenience of machine computation, these bits may be regarded as “signs” on the sequences $LnR$ and $GnG$, if sufficient care is taken to distinguish $-0$ and $+0$. State the recursions which allow one to compute these sequences.
2.5 Col

Recall that in the game of Col (which simplifies the problems of farm management), only L can move onto a dark-tinted node \( \bullet \), and only R can move onto a light-tinted node \( \bigcirc \). Either player can move onto an untinted node, and any move onto a node results in adjacent nodes being tinted with the mover’s color. Occupied nodes and nodes tinted with both colors may be removed from the graph. Each player makes a single move onto any permissible node he wishes at each turn, until the mover is unable to move at which occasion he loses and the game ends.

**Problem:** Determine recursions for the values of all of the following single-string graphs in the game of Col:

(a) \( \bullet \cdot n \cdot \bullet \) which means \( \bullet \cdot n \cdot \bullet \)

(b) \( \bullet \cdot n \cdot \bigcirc \)

(c) \( \bullet \cdot n+1 \)

(d) \( \bigcirc \cdot n+2 \)

???????????????? Solution on following page ?????????????????
Solution: (review page 249)

(a) 
\[ .^n = \left\{ .^k . + .^i \right\} \mid .^k . + .^i = \{2 \mid 0\} = 1. \]

(b) 
\[ .^n = \left\{ .^k . + .^i . \right\} \mid .^k . + .^i = \{0 - 1 \mid 0 + 1\} = 0. \]

(c) 
\[ .^{n+1} = \left\{ .^k . + .^i \right\} \mid .^k . + .^i = \\{0 - \frac{1}{2}, 0 \mid 1\} = \frac{1}{2}. \]

(d) 
\[ .^{n+2} = \left\{ .^k . + .^i \right\} \mid .^k . + .^i = \left\{ -\frac{1}{2} + (-\frac{1}{2}), -\frac{1}{2} \mid \frac{1}{2} + \frac{1}{2}, \frac{1}{2}\right\} = 0. \]
A game of Col

```
L
R
L
R
```
2.6 Seating Couples

A game of Seating Couples
2.7 Atomic Weight Problems

Problems: Determine the atomic weight of each of the following games.

\[
E = \uparrow | \downarrow \\
F = \uparrow | \downarrow \\
G = \uparrow | \downarrow \\
H = \uparrow | \uparrow \\
I = \uparrow | \uparrow \\
J = \uparrow * | \uparrow
\]

Solution on following page
**Solutions:** Value of atomic weights:

<table>
<thead>
<tr>
<th></th>
<th>values</th>
<th>Atomic weights</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>E</strong></td>
<td>$\uparrow \downarrow$</td>
<td>*</td>
</tr>
<tr>
<td><strong>F</strong></td>
<td>$\uparrow \downarrow$</td>
<td>$-\frac{1}{2}$</td>
</tr>
<tr>
<td><strong>G</strong></td>
<td>$\uparrow \downarrow = *$</td>
<td>0</td>
</tr>
<tr>
<td><strong>H</strong></td>
<td>$\uparrow \uparrow = \uparrow *$</td>
<td>2</td>
</tr>
<tr>
<td><strong>I</strong></td>
<td>$\uparrow \uparrow = 3 \cdot \uparrow *$</td>
<td>3</td>
</tr>
<tr>
<td><strong>J</strong></td>
<td>$\uparrow \uparrow = 0$</td>
<td>0</td>
</tr>
</tbody>
</table>
2.8 More Atomic Weight Problems

Problems: For each of the following entire games, determine the value of \( i \) such that the game \( \ast i \uparrow \).

1. The mixed Hackenbush game:

   \[
   \begin{array}{cccc}
   & L & N & R & R \\
L & N & N & N & N \\
   & N & N & N & N \\
   \end{array}
   \]

2. The sheep and goats’ game \( S^2GS - SGSG^5 = \{0|\{0|\{-\frac{1}{8}| - 2\}\}\}\). (Sheep move toward the right, and goats move toward the left.)

3. \( \{0,*,*5 \mid 0,*2,*3,*6\} \)

4. \( \{0 \mid \downarrow\} \)

5. \( \{0 \mid \downarrow +*\} \)

Solution on following page
Solutions:

1. $G = \downarrow$ because Right has 3 paths to ground; Left has only 2.

2. $\{0|\{0|\text{neg}\}\} = 0$ and $\{0|\{0|\{	ext{neg}\}\}\} = \uparrow$.

3. $\uparrow$ - Left can reach * but Right cannot.

4. $\{0| \downarrow\} = * = 0$.

5. $\{0| \downarrow + *\} = 0$. 
3 Ski Jumps

Various games of this type may be played on Cartesian boards of various dimensions. Each player begins the game with an assortment of checkers at appropriate locations. The checkers may be either kings or small men, or some of each. Left plays the Light men, and Right plays the "daRk" men. A legal move for Left is to advance any one of his men as many squares as he wishes from West to East; a legal move for Right is to advance any one of his men as many squares as he wishes from East to West. Men which are advanced horizontally are not allowed to land on or pass over other men of either color, although if there is no one ahead of them, they may move all the way off of the board.

We imagine the board to be situated on a hill with North at the top and South at the bottom. If two men of opposite color occupy the same meridian on adjacent squares, and if the higher man is a king, then, instead of making a latitudinal move, the king may, if he wishes, jump over his opponent, landing on the next square below him on the same meridian. If this square is already occupied, the jump is not possible. If the jumpee was a king, he becomes demoted by the jump to a small man; if he was already a small man, the jumpee is unaffected by the jump. In either case, the jumper remains a king.

The game ends when someone is unable to move, and the player unable to make a legal move loses.

Problem: One interesting game of this type is shown in the following Figure, where the game is being played on three $7 \times 3$ boards. On each board, there is a Light king on the top row, and it threatens eventually to jump the daRk king on the middle row. Left is now to move and win. What is his winning move?

Solution: Let Left be the uphill player, and let Right be the downhill player. Let $x$ be the number of single advancing moves which Left can make until he leaves the board, and let $y$
be the number of moves which Right can make before leaving the board. Let \( g(x, y) \) denote the value of the game we have just described. Evidently, the value of the game shown in the figure above is \( g(7, 4) + g(3, 7) + g(6, 6) \). If the board has length \( n \), then an immediate jump is possible iff \( x + y = n + 1 \). If \( x + y \neq n + 1 \), then each player's best move is to advance one square, so in this case \( g(x, y) = \{g(x - 1, y) \mid g(x, y - 1)\} \). Solving this recursion for \( x + y \leq n \), we start with \( g(0, 0) = 0 \) and deduce that \( g(x, y) = x - y \) for all \( x + y \leq n \). Since the condition that \( x + y \leq n \) is equivalent to the statement that the two men have already passed each other, the solution in this region could also have been obtained by inspection.

We next suppose \( x + y = n + 1 \). The best move for Right is still to advance one square, but the best move for Left is now an immediate jump, which brings him to a position of value \( x - y \). Thus, if \( x + y = n + 1 \), then

\[
g(x, y) = \{x - y \mid g(x, y - 1)\} = \{2x - n - 1 \mid 2x - n - 2\} = \frac{1}{2} + 2x - n - 1 = \frac{1}{2} + x - y.
\]

The values of \( g(x, y) \) for \( x + y > n + 1 \) are then found from the recursion \( g(x, y) = \{g(x - 1, y) \mid g(x, y - 1)\} \).

If we tabulate the values of \( g(x, y) \) in the first quadrant of the \( xy \)-plane, then each value not on the diagonal \( x + y = n + 1 \) is found by evaluating the games whose left successor is the number to the left of the square being evaluated, and whose right successor is the number just below the square being evaluated. The results for the particular case in which \( n = 5 \) are shown in the following table, where the exceptional diagonal is written in small print.

<table>
<thead>
<tr>
<th>( y )</th>
<th>( 5 )</th>
<th>( { -4 } = -5 )</th>
<th>( {-4} - 3 = -6 \frac{1}{2} )</th>
<th>( {-3} - 1 = -2 \frac{1}{2} )</th>
<th>( {-2} 0 = -1 )</th>
<th>( {-1} = 0 )</th>
<th>( {0} 1 = \frac{1}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4 )</td>
<td>( { -3 } = -4 )</td>
<td>( {-4} - 2 = -5 )</td>
<td>( {-2} - 1 = -3 )</td>
<td>( {-1} = 0 )</td>
<td>( {0} = \frac{1}{2} )</td>
<td>( {\frac{1}{2}} = 1 )</td>
<td></td>
</tr>
<tr>
<td>( 3 )</td>
<td>( { -2 } = -3 )</td>
<td>( {-3} - 1 = -4 )</td>
<td>( {-2} 0 = -1 )</td>
<td>( {0} 1 = \frac{1}{2} )</td>
<td>( {\frac{1}{2}} = 1 )</td>
<td>( {1} = 2 )</td>
<td></td>
</tr>
<tr>
<td>( 2 )</td>
<td>( { -1 } = -2 )</td>
<td>( {-2} 0 = -1 )</td>
<td>( {-1} = 0 )</td>
<td>( {0} 1 = \frac{1}{2} )</td>
<td>( {\frac{1}{2}} = 1 )</td>
<td>( {2} = 3 )</td>
<td></td>
</tr>
<tr>
<td>( 1 )</td>
<td>( { 0 } = -1 )</td>
<td>( {-1} = 0 )</td>
<td>( {0} 1 = \frac{1}{2} )</td>
<td>( {1} = 2 )</td>
<td>( {2} = 3 )</td>
<td>( {3} = 4 )</td>
<td></td>
</tr>
<tr>
<td>( 0 )</td>
<td>( { } = 0 )</td>
<td>( {0} = 1 )</td>
<td>( {1} = 2 )</td>
<td>( {2} = 3 )</td>
<td>( {3} = 4 )</td>
<td>( {4} = 5 )</td>
<td></td>
</tr>
</tbody>
</table>

| \( x \) | 0 | 1 | 2 | 3 | 4 | 5 |

Let us define the function \( j(x, y) = g(x, y) - x + y \). We interpret \( j \) as the value of the jump to \( Left \). The value of \( g \) is then given as the value of the jump plus the difference in the relative meridians of the two men on the board. As seen from the above table for the case \( n = 5 \), the value of the jump is given by the following formula:

\[
j(x, y) = \begin{cases} 
0, & \text{if } x + y < n + 1, \\
\frac{1}{2}, & \text{if } x + y = n + 1, \\
\frac{1}{3}, & \text{if } x + y > n + 1 \text{ and } x = y, \\
1, & \text{if } x + y > n + 1 \text{ and } x > y, \\
0, & \text{if } x + y > n + 1 \text{ and } x < y.
\end{cases}
\]

When \( x + y \leq n \), the men have already passed each other and the jump is obviously worthless. When \( x + y = n \), the jump is imminent, and it is not hard to see that in a big
game consisting of the sum of many such cases, Left will be able to take half of the jumps and Right will escape the other half. When \( x + y > n + 1 \), the two men are still approaching each other, and it is in this region that the value of the jump function is most interesting. What the formula says is that if Right is closer to the center of the board, the potential jump is of no advantage to Left, but that if Left is closer to the center, the “potential” jump is a full move advantage, just as if it were a sure thing! When the two men are symmetrically located about the center, the potential jump is worth a half a move advantage, just as if the jump were imminent.

Let us now solve the game shown at the beginning of this section. On the top board, the jump is worth 1 move, and will still be worth 1 even if either player advances one square. On the middle board, the jump is worthless and will remain worthless even if either player advances one square. Evidently, the price of moving on either of these boards is a full move. However, on the bottom board the jump is worth \( \frac{1}{2} \), and a forward move by either player will alter the value of the jump in his favor. Thus, a move on the bottom board costs only a half, and so this is the best board on which to move. By totalling up the relative meridians in addition to the value of the jumps, it is seen that Left playing first can win if he advances his man on the bottom board a single move to the east.

### 3.1 More Ski Jumps

In the game of 1 Light vs 1 daRk on a \( 3 \times n \) board, the two men are engaged in a race for control of the critical column in the center of the board. If R arrives there first, he prevents the jump because he no longer needs to advance until he is attacked. But if L arrives there first, he traps R on his own territory, and R can realize the advantage of his bigger forward distance remaining to be traversed only if he first permits himself to be jumped. If the columns are numbered from 1 to \( n \), corresponding to the number of moves remaining for a Light man located in that column, then it is not hard to see that the “finish line” is \( \left\lceil \frac{n-1}{2} \right\rceil \), by which we denote the least integer greater than or equal to \( \frac{n-1}{2} \).

Next consider the game of several Light men on the top row vs 1 daRk man on the middle row of a \( 3 \times n \) board. As a particular example, let \( n = 23 \), and let there be four Light men, which have 3, 5, 7, and 19 forward moves remaining, respectively, and place the single daRk man on square 1 (meaning that he has 23 forward moves remaining). How many moves advantage is this game to Light? Since Light can make \( 3 + 5 + 7 + 19 = 34 \) forward moves and daRk can make only 23, Light’s advantage is at least 11. On the other hand, daRk can be jumped at most four times, so Light’s advantage is no more than 15. But what is the advantage, precisely?

Before tackling that relatively difficult problem, it is easier to consider certain special cases first. The most instructive special cases of many Light men on the top row vs a single daRk man on the middle row of a \( 3 \times n \) board turn out to be the “blocked” positions, in which neither player can advance any man a single square forward without creating an imminent jump threat. Such a position occurs when Light’s \( k \) men are located at \( z, z + 1, \ldots, z + k - 1 \), and daRk’s man is located at square \( z - 1 \). Suppression the boardlength \( n \), we denote this game by the symbol

\[
[z, z + 1, \ldots, z + k - 1 \mid z - 1].
\]

From this position, daRk can make \( n - z \) forward moves. If Light manages to realize all \( k \)
jumps, then he will be able to use up \(kz + \binom{k}{2} + k\) moves. Hence, if \(kz + \binom{k}{2} + k \leq n + 1 - (z - 1)\), the overall position represents no advantage to Light. If the value of this game is a negative integer, then L cannot be compelled to move in such a game. If R moves, L can jump and R will eventually be compelled to move again, and be jumped again. Not until all \(k\) jumps have been taken is L able to realize the extra moves which are present in his positional advantage. Thus, the blocked position is worth \(k\) jumps to L whenever

\[
kz + \binom{k}{2} + k \leq n - z + 2
\]

or equivalently,

\[
(k + 1)z + \binom{k+1}{2} < n + 3.
\]

Let us now suppose that translating the blocked position over one column destroys daRk’s reason to go first. The L men now occupy \(z+1, z+2, \ldots, z+k\), and the R man now occupies \(z\). Even if he must concede the first jump, \(L\) still has \(kz + \binom{k+1}{2} + k - 1\) moves if he can realize the remaining \(k - 1\) jumps. If this is no less than \(R\)’s \(n + 1 - z\), then the overall position is no advantage to R. If the value of this game is a nonnegative integer, then R can ignore it until L initiates the attack. R can then escape the first jump by responding to L’s attack, but L is then able to use up \(z\) moves while R remains blocked. Assuming that these \(z\) moves are enough to change the game into one no longer advantageous for L, then R can cash in on his remaining surplus of forward moves only if he first concedes \(k - 1\) jumps.

We are thus led to define the \(k\)-th critical, \(z_k\), as the (unique) integral solution of the inequalities

\[
\frac{n + 1}{k+1} - \frac{k}{2} - 1 \leq z_k < \frac{n + 3}{k+1} - \frac{k}{2}.
\]

Making our previous assertions more precise, we now claim that

\[
[z_k, z_k + 1, \ldots, z_k + k - 1 | z_k - 1] = kz + \binom{k}{2} - (n+1) + (z-1) + k
\]

and that

\[
[z_k + 1, z_k + 2, \ldots, z_k + k | z_k] = kz + \binom{k+1}{2} - (n+1) + z + k - 1.
\]

The last term in each equation is the number of jumps. Our assertion is that if R occupies the critical file \(z_k\) in the blocked position, then L gets only \(k - 1\) jumps, but that if L occupies \(z_k\) in the blocked position, then L gets all \(k\) jumps.
Let us now look at the board of length \( n = 23 \). On this board, the critical files are

\[
\begin{align*}
z_1 &= \left\lfloor \frac{26}{2} \right\rfloor - \frac{1}{2} = 12, \\
z_2 &= \left\lfloor \frac{26}{3} \right\rfloor = 7, \\
z_3 &= \left\lfloor \frac{26}{4} \right\rfloor - \frac{1}{2} = 4, \\
z_4 &= \left\lfloor \frac{26}{5} \right\rfloor - 3 = 3,
\end{align*}
\]

\ldots

Let us now play a game starting from the position \([18, 7, 5, 2 | 1]\). Since \( L \) has a large advantage of \( 18 + 7 + 5 + 2 + j - 23 = 9 + j \), where \( j \) appears to be 3 or 4, we will play this game as a summand in a larger game whose other summand is an \( L \)-\( R \) Hackenbush string \( R^{13} \). This string gives \( R \) another option on which to play if he chooses not to move in the ski-Jumps game. If we suppose that \( R \) opens, the first few moves are obvious.

<table>
<thead>
<tr>
<th>Left to</th>
<th>Right to</th>
</tr>
</thead>
<tbody>
<tr>
<td>([18, 7, 5, 2</td>
<td>1])</td>
</tr>
<tr>
<td>([18, 7, 5, 2'</td>
<td>2])</td>
</tr>
</tbody>
</table>

We simplify notation by suppressing the \( L \) men on the bottom row, who are effectively playing in a disjoint game.

If \( L \) now fails to move the man on 5, then \( R \) can reach the critical column \( z_3 = 4 \) before \( L \)'s man on 5. Once there, \( R \) need no longer move: he can play elsewhere (on the Hackenbush string) until attacked. \( L \) may gradually advance his men from 18 and 7 into the blocked position \([7, 6, 5 | 4]\), but since this game is not negative, \( R \) still has no reason to play. Thus, \( L \)'s only hope of getting all remaining jumps is to win the race to the third critical file \( z_3 = 4 \). Our game continues:

| \([18, 7, 4 | 3]\) | \([18, 7, 4 | 4]\) |
|---------------------|---------------------|
| \([18, 7 | 4]\)       | \([18, 7 | 5]\)      |

\( L \) might now be tempted to advance from 7 to 6, but this is a poor move. The finish line for the race is \( z_2 = 7 \), and this race has already been won by \( L \)! \( L \) cannot afford to waste time pushing a man on passed the finish line when there is still another close race in progress.
It is now clear that R cannot avoid the remaining jump. He might stall by playing out the 13 moves on his Hackenbush string, but L can match this by advancing the men on $7'$, $4'$, and $2'$. We conclude that R, moving first, could not win the original game, and hence that

$$[18, 7, 5, 2 | 1] - 13 \geq 0.$$ 

Since Left cannot get more than all four jumps,

$$[18, 7, 5, 2 | 1] = 13.$$ 

Similarly, $[18, 7, 5, 3 | 1] = 14$. On the other hand, in the game $[19, 7, 5, 3 | 2]$, R has too large a head start, and L cannot get more than 3 jumps even if L makes the next move. Thus, $[19, 7, 5, 3 | 2] = 19 + 7 + 5 + 3 + 3 - 22 = 15$. From these results, it is not difficult to see that $[19, 7, 5, 3 | 1] = 14 \frac{1}{2}$. In this game the races are so close that the outcome depends on who moves first, and the potential jumps are worth precisely $3 \frac{1}{2}$ moves to L.

It is convenient to let the term “L’s first man” denote the man who has furthest to go along the top row, even though, in the direction they are moving, this man is actually last. With the convenient terminology, we record the following observation:

The dark man on the middle row may avoid at least one of the potential jumps if, for any $k$, he can beat the $k$-th Light man to the critical file $z_k$. Conversely, if Light can play so as to win all $i$ critical races, then the $i$ potential jumps are worth a full $i$ moves.

If the contest is not about whether L will get all $k$ jumps, but about whether he will get as many as $k - i$ jumps, then the finish lines are $y_{i+1}, y_{i+2}, \ldots$, where

$$y_{k+i} = z_k - i.$$ 

To avoid being jumped by all but one of the Light men ahead of him on the board of length 23, daRk can do any of the following:

(i) Beat the 1st Light man to file 12 (since all but one of one is zero).
(ii) Beat the 2nd Light man to file 11.
(iii) Beat the 3rd Light man to file 6.
(iv) Beat the 4th Light man to file 3.

Light needs to win all of these races in order to get all but one of the remaining jumps.

**Open Problem:** Analyze the games where Light has several men on the top and bottom rows vs daRk men on the middle row of the $3 \times n$ board. Beware of the possibility that some potential jumps may be blocked by Light men on the bottom row. When there is only one daRk man and no Light men on the bottom row, then if the jump is not imminent, it happens that the value of the game is either an integer or a half integer. Is this still true under more general circumstances, with more men on the board?

### 3.2 Even More Ski Jumps

1. We consider ski-jump positions on the $3 \times 5$ board which have two Left skiers and one Right skier. We abbreviate such positions by ommitting the bottom row and denoting empty spaces on the top rows by zeros:

   \[
   \begin{array}{c|c|c}
   L & 0 & L \\
   0 & 0 & R \\
   \end{array}
   \]

   denotes

   \[
   \begin{array}{c|c|c}
   L & L \\
   R \\
   \end{array}
   \]

   Using the results on page 11 of *Winning Ways First Edition*, evaluate each of the following ten positions:

   \[
   \begin{array}{c|c|c}
   L & 0 & 0 & L & 0 \\
   0 & 0 & 0 & R & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   0 & L & 0 & L & 0 \\
   0 & 0 & R & 0 & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   0 & 0 & L & L & 0 \\
   0 & 0 & 0 & R & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   L & 0 & L & 0 & 0 \\
   0 & 0 & 0 & R & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   0 & L & 0 & L & 0 \\
   0 & 0 & R & 0 & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   0 & 0 & L & L & 0 \\
   0 & 0 & 0 & R & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   L & L & 0 & 0 \\
   0 & R & 0 & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   L & 0 & L & 0 & 0 \\
   0 & 0 & R & 0 & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   0 & L & 0 & L & 0 \\
   0 & 0 & 0 & R & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   0 & L & L & 0 & 0 \\
   0 & 0 & R & 0 & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   0 & 0 & L & L & 0 \\
   0 & 0 & 0 & R & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   L & L & 0 & 0 \\
   0 & R & 0 & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   L & L & 0 & 0 \\
   0 & R & 0 & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   0 & L & L & 0 & 0 \\
   0 & R & 0 & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   0 & L & 0 & L & 0 \\
   0 & 0 & R & 0 & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   0 & 0 & L & L & 0 \\
   0 & 0 & 0 & R & 0 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c}
   L & L & 0 & 0 \\
   0 & R & 0 & 0 \\
   \end{array}
   \]

   **Answer:**
2. We seek to evaluate this ski jump position:

\[
\begin{array}{ccc}
& L & L \\
L & 0 & 0 \\
0 & 0 & 0 & R
\end{array}
\]

For notational convenience only, we may omit the bottom row and denote this by

\[
\begin{array}{ccc}
L & 0 & 0 \\
0 & 0 & 0 & R
\end{array}
\]

which appears at the top of the attached tree of positions.

Each of the bold boxed positions has an imminent jump.

(a) Using the results of problem 1, assign a value to each bold boxed position.

(b) Evaluate the other twelve positions shown in the tree.

**Answer:**

\[
\begin{array}{c}
L 0 0 L = \frac{1}{2} + 1 | 1 + 1 = 1 \frac{1}{2} | 2 = 1 \frac{3}{4}.
\\
0 L 0 L = 0 + 1 | \frac{1}{2} + 1 = 1 | 1 \frac{1}{2} = 1 \frac{1}{4}.
\\
0 0 L 0 L = -1 + 1 | 0 + 1 = 0 | 1 = \frac{1}{2}.
\\
0 0 0 L L = -2 + 1 | -1 + 1 = -1 | 0 = -\frac{1}{2}.
\end{array}
\]
\[
\begin{array}{ccc|ccc}
L & 0 & 0 & L & 0 & 0 \\
0 & 0 & 0 & R & 0 & 0 \\
& & & & & \\
= & 1 + 2 & | & 2 + 2 & = & 3 \mid 4 = 3\frac{1}{2}.
\end{array}
\]

\[
\begin{array}{ccc|ccc}
0 & L & 0 & L & 0 & 0 \\
0 & 0 & 0 & R & 0 & 0 \\
& & & & & \\
= & \frac{1}{2} + 2 & | & 1 + 2 & = & 2\frac{1}{2} \mid 3 = 2\frac{3}{4}.
\end{array}
\]

\[
\begin{array}{ccc|ccc}
0 & 0 & L & L & 0 & 0 \\
0 & 0 & 0 & R & 0 & 0 \\
& & & & & \\
= & 0 + 2 & | & \frac{1}{2} + 2 & = & 2 \mid 2\frac{1}{2} = 2\frac{1}{4}.
\end{array}
\]

\[
\begin{array}{ccc|ccc}
L & 0 & L & 0 & 0 & 0 \\
0 & 0 & R & 0 & 0 & 0 \\
& & & & & \\
= & 2 + 3 & | & 3 + 3 & = & 5 \mid 6 = 5\frac{1}{2}.
\end{array}
\]

\[
\begin{array}{ccc|ccc}
0 & L & L & 0 & 0 & 0 \\
0 & 0 & R & 0 & 0 & 0 \\
& & & & & \\
= & 1 + 3 & | & 3\frac{1}{2} + 3 & = & 4 \mid 5\frac{1}{2} = 5.
\end{array}
\]

\[
\begin{array}{ccc|ccc}
L & L & 0 & 0 & 0 & 0 \\
0 & R & 0 & 0 & 0 & 0 \\
& & & & & \\
= & 3 + 4 & | & 4\frac{1}{2} + 4 & = & 7 \mid 8\frac{1}{2} = 8.
\end{array}
\]
3. This problem first appeared April 6, 1982, in a Math 195 homework due on April 15.

$$g_A = \begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\end{array}$$

$$g_B = \begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\end{array}$$

**Hint**: When are the following two games equal?

$$\begin{array}{cccccccccccc}
& & L & k & R \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
\end{array}$$

$$\begin{array}{cccccccccccc}
& & L & k & R \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
\end{array}$$

4. Say as much as you can about the value of the following position on the 100-column ski jump board: Left has men at 5, 15, 25, 35, 45, 55, 65, 75, 85 and 95; Right starts at 95. All Left’s are on top row; Right is in the middle.

**Answer:**

$$\begin{array}{cccccccccccc}
& & L & j & R \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
& & \ \ & \ \ & \ \ \\
\end{array}$$

\[ (B - i - j) + \frac{j(j+1)}{2} - i - j - 1, \text{ denoted } N_B(i, j). \]

\[
\begin{align*}
N_B(i, j) + j, & \quad \text{if this is } \geq 0, \\
\leq N_B(i, j) + j - 1, & \quad \text{if this is } > 0.
\end{align*}
\]

So the “jth finish line” is the value of i + j at which N_B(i, j) + j changes sign. Call this f (for finish line):

\[
(B - f) + \frac{j(j+1)}{2} - f - 1 + j = 0 \text{ or } 1
\]

\[
f(j + 1) = \begin{cases} 
\frac{j(j+1)}{2} + j + Bj - 1, & \text{or} \\
\frac{j(j+1)}{2} + j + Bj - 2.
\end{cases}
\]

f = \frac{j}{2} + \frac{j + Bj - 1}{j+1} \text{ or } \frac{j}{2} + \frac{j + Bj - 2}{j+1}. \text{ Set } B = 30, \text{ consider only } f = \frac{j}{2} + \frac{j + Bj - 1}{j+1}:

<table>
<thead>
<tr>
<th>j</th>
<th>f</th>
<th>ALL but 1</th>
<th>ALL but 2</th>
<th>ALL but 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15\frac{1}{7}</td>
<td>15</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>21\frac{2}{3}</td>
<td>21</td>
<td>22</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>24\frac{3}{7}</td>
<td>24\frac{3}{7}</td>
<td>25</td>
<td>23</td>
</tr>
<tr>
<td>4</td>
<td>26\frac{4}{7}</td>
<td>26\frac{4}{7}</td>
<td>27</td>
<td>26</td>
</tr>
<tr>
<td>5</td>
<td>28\frac{5}{7}</td>
<td>28</td>
<td>29</td>
<td>28</td>
</tr>
<tr>
<td>6</td>
<td>29\frac{6}{7}</td>
<td>29\frac{6}{7}</td>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>
In order to get ALL jumps, Left’s 5th man must reach 29 before Right, his 4th man must reach 27 before Right, etc. To get all but one, Left’s 5th man must reach 28; his 4th man must reach 26, etc.

The real contest in $g_A$ is for Left to get all but 2 jumps. The 5th man is already on 27, so the races finish at 24 and 18 respectively. If Left moves 1st, he wins both races and gets all but 2 of the jumps ($J = 3$). If Right moves 1st, R wins one of the races and Left gets only 2 jumps. So $J = 2 \frac{1}{2}$ and $g_A = 54 + 2 \frac{1}{2} = 56 \frac{1}{2}$.

In $g_B$, Left seeks to get 3 jumps by reaching 28, 25 and 19. To do this he needs:

<table>
<thead>
<tr>
<th>Goal</th>
<th>Moves</th>
<th>+Jumps</th>
<th>Total</th>
<th>Right needs</th>
</tr>
</thead>
<tbody>
<tr>
<td>28</td>
<td>1</td>
<td></td>
<td>1</td>
<td>30-28 = 2</td>
</tr>
<tr>
<td>25</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td>30-25 = 5</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td></td>
<td>8</td>
<td>30-19 = 11</td>
</tr>
</tbody>
</table>

Left going first gets his 3 jumps, but Right going first holds him to 2 jumps by winning the race to 28 or to 25. Again, $J_B = 2 \frac{1}{2}$ and $g_B = (31 - 27) + (31 - 22) + (31 - 19) + (31 - 12) + (31 - 7) + (31 - 2) - 30 + 2 \frac{1}{2} = 69 \frac{1}{2}$.

Finally, take $B = 100$ so $f = \frac{j}{2} + \frac{10^j-1}{j+1}$.

| $j$ | $|f|$ | all but 1 | all but 2 | all but 3 | all but 4 | all but 5 | Start |
|-----|-------|-----------|-----------|-----------|-----------|-----------|-------|
| 1   | 51    | 52        | 69        | 78        | 84        | 97        | 5     |
| 2   | 68    | (no rounding!) | 52        | 69        | 78        | 84        | 15    |
| 3   | 77    | (no rounding!) | 69        | 78        | 84        | 97        | 25    |
| 4   | 83    | 88        | 93        | 95        | 97        | 96        | 35    |
| 5   | 87    | 88        | 93        | 95        | 97        | 96        | 45    |
| 6   | 90    | 88        | 93        | 95        | 97        | 96        | 55    |
| 7   | 92    | 88        | 93        | 95        | 97        | 96        | 65    |
| 8   | 94    | 88        | 93        | 95        | 97        | 96        | 75    |
| 9   | 96    | 88        | 93        | 95        | 97        | 96        | 85    |
| 10  | 97    | 88        | 93        | 95        | 97        | 96        | 95    |
So Left gets 5 jumps if he starts, 4 otherwise. If Right starts, he avoids the jump at 95 and wins a race at 91 or 82.

### Solution to 30-square board Ski Jump

In game A, Left strives to move his men:

\[
\begin{array}{c|c|c|c|c}
\text{from} & \text{to} \\
21 & 24 \\
11 & 18 \\
\end{array}
\]

before Right gets to the corresponding squares.

In game B, Left strives to move

\[
\begin{array}{c|c|c|c|c}
\text{from} & \text{to} \\
27 & 28 \\
22 & 25 \\
17 & 19 \\
\end{array}
\]

By achieving these objectives, Left ensures jumps with all-but-2 in A, or with all-but-3 in B. In each case, this means at least 3 jumps if Left goes first.

If Right thwarts ANY of Left’s objectives, by arriving at a “to” square before the corresponding Left skier, he patiently leaves the skier alone, and never moves it again except when directly attacked or when he has run out of Hackenbush branches. This strategy ensures at most 2 jumps if Right goes first.
(Detailed calculations of the “to” finish lines are given below.)

\[ 15 = 15 \]

\[ 19 < 11 + 10, \]
\[ 20 > 10 + 9, \]

\[ 21 < 9 + 8 + 7, \]
\[ 22 > 8 + 7 + 6, \]

\[ 22 < 8 + 7 + 6 + 5 \]
\[ 23 > 7 + 6 + 5 + 4 \]

\[ 23 < 7 + 6 + 5 + 4 + 3 \]
\[ 24 > 6 + 5 + 4 + 3 + 2 \]

<table>
<thead>
<tr>
<th>Starting position of A</th>
<th>3</th>
<th>11</th>
<th>21</th>
<th>27</th>
</tr>
</thead>
<tbody>
<tr>
<td>Starting position of B</td>
<td>2</td>
<td>7</td>
<td>12</td>
<td>17</td>
</tr>
<tr>
<td>1st</td>
<td>15</td>
<td>20</td>
<td>22</td>
<td>23</td>
</tr>
<tr>
<td>2nd</td>
<td>16</td>
<td>22</td>
<td>25</td>
<td>27</td>
</tr>
<tr>
<td>3rd</td>
<td>17</td>
<td>23</td>
<td>26</td>
<td>28</td>
</tr>
<tr>
<td>4th</td>
<td>18</td>
<td>24</td>
<td>27</td>
<td>29</td>
</tr>
<tr>
<td>5th</td>
<td>19</td>
<td>25</td>
<td>28</td>
<td>\infty</td>
</tr>
<tr>
<td>6th</td>
<td>20</td>
<td>26</td>
<td>\infty</td>
<td>\infty</td>
</tr>
</tbody>
</table>

Left, going first, gets ≥ 3 jumps.
Right, going first, lowers to ≤ 2 jumps.

\[ g_A = 28 + 22 + 20 + 10 + 4 - 30 + 2\frac{1}{2} = 56\frac{1}{2}, \]
\[ g_B = 29 + 24 + 19 + 14 + 9 + 4 - 30 + 2\frac{1}{2} = 71\frac{1}{2}. \]
Next, we compute for the 100-column board.

\[
50 = 50, \\
66 < 34 + 33, \\
67 > 33 + 32, \\
74 < 26 + 25 + 24, \\
75 > 25 + 24 + 23, \\
78 < 22 + 21 + 20 + 19, \\
79 > 21 + 20 + 19 + 18, \\
81 < 19 + 18 + 17 + 16 + 15, \\
82 > 18 + 17 + 16 + 15 + 14, \\
83 < 17 + 16 + 15 + 14 + 13 + 12, \\
84 > 16 + 15 + 14 + 13 + 12 + 11, \\
84 < 16 + 15 + 14 + 13 + 12 + 11 + 10, \\
85 > 15 + 14 + 13 + 12 + 11 + 10 + 9, \\
85 < 15 + 14 + 13 + 12 + 11 + 10 + 9 + 8, \\
86 > 14 + 13 + 12 + 11 + 10 + 9 + 8 + 7, \\
86 < 14 + 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6, \\
87 > 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5, \\
86 < 14 + 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5, \\
87 > 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4, \\
87 < 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3, \\
88 > 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1.
\]
The 100-column board is numbered 1 thru 100. Left starts with 10 skiers on 5, 15, 25, \ldots, 95 on the top row. Right’s lone skier begins in middle row, column 97. Hence, \( g = 500 - 97 + \frac{41}{2} = 407 \frac{1}{2}. \)