Kac-Moody Algebras and Applications

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Abstract

This article is an introduction to the theory of Kac-Moody algebras: their genesis, their construction, basic theorems concerning them, and some of their applications. We first record some of the classical theory, since the Kac-Moody construction generalizes the theory of simple finite-dimensional Lie algebras in a closely analogous way. We then introduce the construction and properties of Kac-Moody algebras with an eye to drawing natural connections to the classical theory. Last, we discuss some physical applications of Kac-Moody algebras, including the Sugawara and Virosoro coset constructions, which are basic to conformal field theory.

Contents

1 Introduction

Élie Cartan, in his 1894 thesis, completed the classification of the finite-dimensional simple Lie algebras over $\mathbf{C}$. Cartan's thesis is a convenient marker for the start of over a half-century of research on the structure and representation theory of finite-dimensional simple complex Lie algebras. This effort, marshalled into existence by Lie, Killing, Cartan, and Weyl, was continued and extended by the work of Chevalley, Harish-Chandra, and Jacobson, and culminated in Serre's Theorem [Ser66], which gives a presentation for all finite-dimensional simple complex Lie algebras.

In the mid-1960s, Robert Moody and Victor Kac independently introduced a collection of Lie algebras that did not belong to Cartan's classification. Moody, at the University of Toronto, constructed Lie algebras from a generalization of the Cartan matrix, emphasizing the algebras that would come to be known as *affine* Kac-Moody Lie algebras, while Kac, at Moscow State University, began by studying simple graded Lie algebras of finite growth. In this way, Moody and Kac independently introduced a new class of generally infinite-dimensional, almost always simple Lie algebras that would come to be known as Kac-Moody Lie algebras.

The first major application of the infinite-dimensional Lie algebras of Kac and Moody came in 1972 with Macdonald's 1972 discovery [Mac71] of an affine analogue to Weyl's denominator formula. Specializing Macdonald's formula to particular affine root systems yielded previously-known identities for modular forms such as the Dedekind η-function and the Ramanujan τ-function. The arithmetic connection Macdonald established to affine Kac-Moody algebras was subsequently clarified and explicated independently by both Kac and Moody, but only marked the beginning of the applications of these infinite-dimensional Lie algebras. It didn't take long for the importance of Kac-Moody algebras to be recognized, or for many applications of them to both pure mathematics and physics to be discovered.

We will structure this article as follows. We will first recall some basic results in the theory of finite-dimensional Lie algebras so as to ease the introduction of their infinite-dimensional relatives. Next, we will define what is a Kac-Moody algebra, and present some fundamental theorems concerning these algebras. With the basic theory established, we will turn to some applications of Kac-Moody algebras to physics.

We include proofs for none of the results quoted in this review, but we do take care to cite the references where appropriate proofs may be found.

2 Finite-dimensional Lie algebras

We collect in this section some basic results pertaining to the structure and representation theory of finite-dimensional Lie algebras. This is the classical theory due to Lie, Killing, Cartan, Chevalley, Harish-Chandra, and Jacobson. The justification for reproducing the results in the finite-dimensional case is that both Moody and Kac worked initially to generalize the methods of Harish-Chandra and Chevalley as presented by Jacobson in [Jac62, Chapter 7]. For this reason, the general outline of the approach to studying the structure of Kac-Moody algebras resembles that of the approach to studying the finite-dimensional case.

We follow the presentation of Serre [Ser00] in this section. We suppose throughout this section that $\mathfrak{g}$ is a *finite-dimensional* Lie algebra over the field of complex numbers, though we will omit both 'finite-dimensional' and 'complex' in the statements that follow. The Lie bracket of x and y is denoted by $[x, y]$ and the map $y \mapsto [x, y]$ is denoted $\operatorname{ad} x$. For basic definitions such as that of an ideal of a Lie algebra (and for proofs of some statements given without proof in [Ser00]), c.f. Humphreys [Hum72].

2.1 Nilpotency

The *lower central series* of $\mathfrak{g}$ is the descending series $(C^n\mathfrak{g})_{n\geqslant 1}$ of ideals of $\mathfrak{g}$ defined by the formulæ

$$C^1\mathfrak{g} = \mathfrak{g} \tag{2.1}$$

$$C^n\mathfrak{g} = [\mathfrak{g}, C^{n-1}\mathfrak{g}] \quad \text{if } n \geqslant 2. \tag{2.2}$$

Definition. *A Lie algebra $\mathfrak{g}$ is said to be* nilpotent *if there exists an integer n such that $C^n\mathfrak{g} = 0$. $\mathfrak{g}$ is nilpotent of class r if $C^{r+1}\mathfrak{g} = 0$.*

Proposition 2.1. *The following conditions are equivalent.*

(i) *$\mathfrak{g}$ is nilpotent of class $\leqslant r$.*

(ii) *There is a descending series of ideals*

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_r = 0 \tag{2.3}$$

such that $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$ for $0 \leqslant i \leqslant r-1$.

Recall that the *center* of a Lie algebra is the set of $x \in \mathfrak{g}$ such that $[x, y] = 0$ for all $y \in \mathfrak{g}$. It is an abelian ideal of $\mathfrak{g}$. We denote the center of $\mathfrak{g}$ by $Z(\mathfrak{g})$.

Proposition 2.2. *Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{a} \subset Z(\mathfrak{g})$ be an ideal. Then*

$$\mathfrak{g} \text{ is nilpotent } \Leftrightarrow \mathfrak{g}/\mathfrak{a} \text{ is nilpotent.} \tag{2.4}$$

We now present the basic results pertaining to nilpotent Lie algebras.

Theorem 2.3 (Engel)**.** *For a Lie algebra $\mathfrak{g}$ to be nilpotent, it is necessary and sufficient for $\operatorname{ad} x$ to be nilpotent $\forall x \in \mathfrak{g}$.*

Theorem 2.4. *Let $\phi : \mathfrak{g} \to \operatorname{End}(V)$ be a linear representation of a Lie algebra $\mathfrak{g}$ on a nonzero finite-dimensional vector space V. Suppose that $\phi(x)$ is nilpotent $\forall x \in \mathfrak{g}$. Then $\exists 0 \neq v \in V$ which is invariant under $\mathfrak{g}$.*

2.2 Solvability

The *derived series* of $\mathfrak{g}$ is the descending series $(D^n\mathfrak{g})_{n\geqslant 1}$ of ideals of $\mathfrak{g}$ defined by the formulæ

$$D^1\mathfrak{g} = \mathfrak{g} \tag{2.5}$$

$$D^n\mathfrak{g} = [D^{n-1}\mathfrak{g}, D^{n-1}\mathfrak{g}] \quad \text{if } n \geqslant 2. \tag{2.6}$$

Definition. *A Lie algebra $\mathfrak{g}$ is said to be solvable if $\exists n \in \mathbf{N}$ s.t. $D^n\mathfrak{g} = 0$. $\mathfrak{g}$ is solvable of derived length $\leqslant r$ if $D^{r+1}\mathfrak{g} = 0$.*

N.B. Every nilpotent algebra is solvable.

Proposition 2.5. *The following conditions are equivalent.*

(i) *$\mathfrak{g}$ is solvable of derived length $\leqslant r$.*

(ii) *There is a descending central series of ideals of $\mathfrak{g}$:*

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_r = 0 \tag{2.7}$$

such that $[\mathfrak{a}_i, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$ for $0 \leqslant i \leqslant r-1$ ($\Rightarrow$ successive quotients $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ are abelian).

In analogy with Engel's theorem, we now record Lie's theorem.

Theorem 2.6 (Lie)**.** *Let $\phi : \mathfrak{g} \to \operatorname{End}(V)$ be a finite-dimensional linear representation of $\mathfrak{g}$. If $\mathfrak{g}$ is solvable, there is a flag D of V such that $\phi(\mathfrak{g}) \subset \mathfrak{b}(D)$, where $\mathfrak{b}(D)$ is the Borel algebra of D (the subalgebra of $\operatorname{End} V$ that stabilizes the flag).*

Theorem 2.7. *If $\mathfrak{g}$ is solvable, all simple $\mathfrak{g}$-modules are one-dimensional.*

Theorem 2.7 leads us to conclude that the representation theory of solvable Lie algebras is very boring. The following section will make this statement more transparent.

2.3 Semisimplicity

Recall that the radical $\operatorname{rad}\mathfrak{g}$ is largest solvable ideal in $\mathfrak{g}$. $\operatorname{rad}\mathfrak{g}$ exists because if $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals of $\mathfrak{g}$, the ideal $\mathfrak{a}+\mathfrak{b}$ is also solvable since it is an extension of $\mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b}) = (\mathfrak{a}+\mathfrak{b})/\mathfrak{a}$ by $\mathfrak{a}$.

Theorem 2.8. *Let $\mathfrak{g}$ be a Lie algebra.*

(a) $\operatorname{rad}(\mathfrak{g}/\operatorname{rad}\mathfrak{g}) = 0$.

(b) *There is a Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$ that is a complement for $\operatorname{rad}\mathfrak{g}$; i.e. such that the projection $\mathfrak{s} \to \mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is an isomorphism.*

In this way, we may easily strip away the radical of a Lie algebra $\mathfrak{g}$ and are left with a Lie algebra with no nontrivial solvable ideal.

Definition. *A Lie algebra $\mathfrak{g}$ is* semisimple *if* rad $\mathfrak{g} = 0$.

Thus Theorem 2.8 says that any Lie algebra $\mathfrak{g}$ is a semidirect product of a semisimple algebra and a solvable ideal (the so-called 'Levi decomposition' of $\mathfrak{g}$). The above justifies the restriction of study to semisimple Lie algebras. The following definition and theorem imply that we may restrict our concern even further.

Definition. *A Lie algebra $\mathfrak{s}$ is said to be* simple *if*

(a) it is not abelian,

(b) its only ideals are 0 and $\mathfrak{s}$.

Theorem 2.9. *A Lie algebra $\mathfrak{g}$ is semisimple iff it is isomorphic to a product of simple algebras. Indeed, let $\mathfrak{g}$ be a semisimple Lie algebra and $(\mathfrak{a}_i)$ be its minimal nonzero ideals. The ideals $\mathfrak{a}_i$ are simple Lie algebras, and $\mathfrak{g}$ can be identified with their product.*

Theorem 2.9, together with the above discussion, shows that the Cartan classification of simple complex finite-dimensional Lie algebras allows us to classify all finite-dimensional Lie algebras. Hermann Weyl proved, via his 'unitarian trick,' the following theorem about representations of semisimple algebras.

Theorem 2.10. *Every finite-dimensional linear representation of a semisimple algebra is completely reducible.*

2.4 Root systems

Before turning to the structure theory of complex semisimple Lie algebras, we need to establish some basic definitions and results pertaining to symmetries of vector spaces. This theory contains the concrete ideas (most notably the Cartan matrix) that are generalized in the formulation of Kac-Moody algebras, since, as we will see in section 2.5, the theory of symmetries of vector spaces contains the information about the decomposition of complex semisimple Lie algebras.

2.4.1 Symmetries

We begin by introducing the notion of a real root system, which, as we will see, is all we need to study the complex root systems that arise in the structure theory of semisimple complex Lie algebras. Let V be a (real) vector space and $0 \neq \alpha \in V$. We begin by defining a *symmetry* with vector α to be any $s \in \text{Aut}\, V$ that satisfies the conditions

(i) $s(\alpha) = -\alpha$.

(ii) The set $H^s \subset V$ of s-invariant elements is a hyperplane of V.

The set H^s is actually a complement for the line $\mathbf{R}\alpha$, and s is an automorphism of order 2. The symmetry is totally determined by the choice of $\mathbf{R}\alpha$ and of H^s. If we denote the dual space of V by V^* and let α^* be the unique element of V^* which vanishes on H^s and takes the value 2 on α. Then

$$s(x) = x - \langle \alpha^*, x \rangle \alpha \quad \forall x \in V. \tag{2.8}$$

Since End $V \simeq V^* \otimes V$, we may instead write

$$s = 1 - \alpha^* \otimes \alpha. \tag{2.9}$$

We make the following

Definition. *Let V be a finite-dimensional vector space. A subset $R \subset V$ is called a root system if*

(1) *R is finite, spans V, and does not contain 0.*

(2) *$\forall \alpha \in R$ there is a unique symmetry $s_\alpha = 1 - \alpha^* \otimes \alpha$ with vector α which leaves R invariant.*

(3) *If $\alpha, \beta \in R$, $s_\alpha(\beta) - \beta$ is an integer multiple of α.*

We say the *rank* of R is dim V and the elements of R are called the *roots* of V (relative to R). We say R is *reduced* if $\forall \alpha \in R$, α and $-\alpha$ are the only roots proportional to α. The element $\alpha^* \in V^*$ as in (2.9) is called the *inverse root* of α.

2.4.2 The Weyl group

Definition. *Let R be a root system in a vector space V. The Weyl group of R is the subgroup $W \subset GL(V)$ generated by the symmetries s_α, $\alpha \in R$.*

Given any positive definite symmetric bilinear form on V, since W is finite we may average it over W's action on V to obtain a W-invariant form.

Proposition 2.11. *Let R be a root system in V. There is a positive definite symmetric bilinear form $(\ ,\ )$ on V invariant under the Weyl group W of R.*

The choice of $(\ ,\ )$ gives V a Euclidean structure on which W acts by orthogonal transformations. We may re-write s_α as

$$s_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha \quad \forall x \in V \tag{2.10}$$

and introduce the notion of *relative position* of two roots $\alpha, \beta \in R$ by the formula

$$n(\beta,\alpha) = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}. \tag{2.11}$$

2.4.3 Bases

Definition. *A subset $S \subset R$ is called a* base *for R if*

(i) *S is a basis for the vector space V.*

(ii) *Each $\beta \in R$ can be written as a linear combination*

$$\beta = \sum_{\alpha \in S} m_\alpha \alpha, \tag{2.12}$$

where the coefficients m_α are integers with the same sign (i.e. all non-negative or all non-positive).

The elements of S are also sometimes called 'simple roots.' Every root system admits a base. Letting R^+ denote the set of roots which are linear combinations with non-negative integer coefficients of elements of S, we say an element of R^+ is a *positive root* (with respect to S).

Proposition 2.12. *Every positive root β can be written as*

$$\beta = \alpha_1 + \cdots + \alpha_k \quad \textit{with } \alpha_i \in S, \tag{2.13}$$

in such a way that the partial sums

$$\alpha_1 + \cdots + \alpha_h, \quad 1 \leqslant h \leqslant k, \tag{2.14}$$

are all roots.

Theorem 2.13. *Suppose* R *is reduced and let* W *be the Weyl group of* R.

(a) $\forall t \in V^* \exists w \in W : \langle w(t), \alpha \rangle \geqslant 0 \forall \alpha \in S$.

(b) *If* S' *is a base for* R, $\exists w \in W$ *such that* $w(S') = S$.

(c) *For each* $\beta \in R$, $\exists w \in W$ *such that* $w(\beta) \in S$.

(d) *The group* W *is generated by the symmetries* s_α, $\alpha \in S$.

2.4.4 The Cartan matrix

We are now prepared to make the following definition, which is the jumping-off point for the study of Kac-Moody algebras.

Definition. *The* Cartan matrix *of* R *(with respect to a fixed base* S*) is the matrix* $(n(\alpha, \beta))_{\alpha,\beta \in S}$.

The following proposition shows that the Cartan matrix encodes all the information contained in a reduced root system.

Proposition 2.14. *A reduced root system is determined up to isomorphism by its Cartan matrix. That is, let* R' *be a reduced root system in a vector space* V'*, let* S' *be a base for* R'*, and let* $\phi : S \to S'$ *be a bijection such that* $n(\phi(\alpha), \phi(\beta)) = n(\alpha, \beta)$ *for all* $\alpha, \beta \in S$*. If* R *is reduced, then there is an isomorphism* $f : V \to V'$ *which is an extension of* ϕ *and maps* R *onto* R'.

That is, any bijection of two bases that preserves the Cartan matrix can be extended linearly to a vector space isomorphism.

2.4.5 Irreducibility

Proposition 2.15. *Suppose that* $V = V_1 \oplus V_2$ *and that* $R \subset V_1 \cup V_2$*. Put* $R_i := R \cap V_i$*. Then*

(a) V_1 *and* V_2 *are orthogonal.*

(b) R_i *is a root system in* V_i.

One says that R is the *sum* of the subsystems R_i. If this can happen only trivially (that is, with V_1 or V_2 equal to 0), and if $V \neq 0$, then R is said to be *irreducible*.

Proposition 2.16. *Every root system is a sum of irreducible systems.*

2.4.6 Complex root systems

The definition of complex root system is identical to the definition in the real case after replacing the real vector space V with a complex one. Actually, it is a theorem that every complex root system may be obtained by extension of scalars of some real root system. Namely, if R is a root system in a complex vector space V, then we may form the $\mathbf{R}$-subspace $V_0 \subset V$. Then R is a (real) root system in V_0, the canonical mapping $V_0 \otimes \mathbf{C} \to V$ is an isomorphism, and $\forall \alpha \in R$ the symmetry s_α of V is a linear extension of the corresponding symmetry in V_0. This shows that the theory of complex root systems reduces totally to the real theory.

2.5 The structure of semisimple Lie algebras

We now turn to the structure theory of semisimple complex Lie algebras. As throughout this section, $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbf{C}$.

2.5.1 Cartan subalgebras

Let $\mathfrak{g}$ be a Lie algebra (not necessarily semisimple) and $\mathfrak{a} \subset \mathfrak{g}$ a subalgebra. Recall that the *normalizer* of $\mathfrak{a}$ in $\mathfrak{g}$ is defined to be the set $\mathfrak{n}(\mathfrak{a})$ of all $x \in \mathfrak{g}$ such that $\mathrm{ad}(x)(\mathfrak{a}) \subset \mathfrak{a}$; it is the largest subalgebra of $\mathfrak{g}$ that contains $\mathfrak{a}$ and in which $\mathfrak{a}$ is an ideal.

Definition. *A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a* Cartan subalgebra *of $\mathfrak{g}$ if it satisfies the following two conditions.*

(a) *$\mathfrak{h}$ is nilpotent.*

(b) $\mathfrak{h} = \mathfrak{n}(\mathfrak{h})$.

If $\mathfrak{h}$ is a Cartan subalgebra of a *semisimple* Lie algebra $\mathfrak{g}$, then $\mathfrak{h}$ is abelian and the centralizer of $\mathfrak{h}$ is $\mathfrak{h}$. It follows that $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$.

2.5.2 Decomposition of $\mathfrak{g}$

For the rest of this section, $\mathfrak{g}$ is assumed to be semisimple and $\mathfrak{h}$ is assumed to be a Cartan subalgebra of $\mathfrak{g}$.

If $\alpha \in \mathfrak{h}^*$, we let $\mathfrak{g}^\alpha$ be the eigensubspace of $\mathfrak{g}$; i.e.

$$\mathfrak{g}^\alpha := \{x \in \mathfrak{g} : [H, x] = \alpha(H)x \forall H \in \mathfrak{h}\}. \tag{2.15}$$

An element of $\mathfrak{g}^\alpha$ is said to have *weight* α. In particular, $\mathfrak{g}^0$ is the set of elements in $\mathfrak{g}$ that commute with $\mathfrak{h}$; since $\mathfrak{g}$ is semisimple, $\mathfrak{h}$ is maximal abelian subalgebra of $\mathfrak{g}$ and so actually $\mathfrak{g}^0 = \mathfrak{h}$.

Theorem 2.17. *One has* $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha$.

Any element $0 \neq \alpha \in \mathfrak{h}^*$ s.t. $\mathfrak{g}^\alpha \neq 0$ is called a *root* of $\mathfrak{g}$ (relative to $\mathfrak{h}$); the set of roots will be denoted R. This suggestive terminology is not a coincidence.

Theorem 2.18. *(a)* *R is a complex reduced root system in $\mathfrak{h}^*$.*

(b) *Let $\alpha \in R$. Then* $\dim \mathfrak{g}^\alpha = 1 = \dim \mathfrak{h}_\alpha$, *where* $\mathfrak{h}_\alpha := [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$. $\exists! H_\alpha \in \mathfrak{h}_\alpha$ *s.t.* $\alpha(H_\alpha) = 2$*; it is the inverse root of α.*

(c) *The subspaces $\mathfrak{g}^\alpha$ and $\mathfrak{g}^\beta$ are orthogonal if $\alpha + \beta \neq 0$. The subspaces $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ are dual with respect to $(\ ,\ )$. The restriction of $(\ ,\ )$ to $\mathfrak{h}$ is nondegenerate.*

2.5.3 Existence and uniqueness

We conclude our remarks on the structure of complex semisimple Lie algebras with the following theorems, which will make clear the significance of the root system (encoded in the Cartan matrix) associated to the Cartan decomposition of $\mathfrak{g}$.

Theorem 2.19 (Existence)**.** *Let R be a reduced root system. There exists a semisimple Lie algebra $\mathfrak{g}$ whose root system is isomorphic to R.*

Theorem 2.20 (Uniqueness)**.** *Two semisimple Lie algebras corresponding to isomorphic root systems are isomorphic. More precisely, let $\mathfrak{g}, \mathfrak{g}'$ be semismple Lie algebras, $\mathfrak{h}, \mathfrak{h}'$ respective Cartan subalgebras of $\mathfrak{g}, \mathfrak{g}'$, S, S' bases for the corresponding root systems, and $r : S \to S'$ a bijection sending the Cartan matrix of S to that of S'. For each $i \in S, j \in S'$, let X_i, X_i' be nonzero elements of $\mathfrak{g}^i$ and $\mathfrak{g}'^j$, respectively. Then there is a unique isomorphism of Lie algebras $f : \mathfrak{g} \to \mathfrak{g}'$ sending H_i to $H'_{r(i)}$ and X_i to $X'_{r(i)}$ for all $i \in S$.*

2.6 Linear representations of complex semisimple Lie algebras

After having described the structure of a complex semisimple Lie algebra $\mathfrak{g}$, we now pass to its linear representations. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$ and R the corresponding root system. We fix a base $S = \{\alpha_1, \ldots, \alpha_n\}$ of R and denote by R^+ the set of positive roots with respect to S. We recall the definition of H_α, the inverse root of α.

Theorem 2.21. *$\forall \alpha \in R \exists X_\alpha \in \mathfrak{g}^\alpha$ such that*

$$[X_\alpha, X_{-\alpha}] = H_\alpha \quad \forall \alpha \in R. \tag{2.16}$$

Fix such $X_\alpha \in \mathfrak{g}^\alpha$, $Y_\alpha \in \mathfrak{g}^{-\alpha}$ so that $[X_\alpha, Y_\alpha] = H_\alpha$. When α is one of the simple roots α_i, we write X_i, Y_i, H_i instead of X_{α_i}, etc. We put $\mathfrak{n} := \sum_{\alpha>0} \mathfrak{g}^\alpha$, $\mathfrak{n}^- := \sum_{\alpha<0} \mathfrak{g}^\alpha$, and $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$.

2.6.1 Weights and primitive elements

Let V be a $\mathfrak{g}$-module of possibly infinite dimension and let $\omega \in \mathfrak{h}^*$ be a linear form on $\mathfrak{h}$. Let V^ω denote the vector subspace of all $v \in V$ such that $Hv = \omega(H)v \forall H \in \mathfrak{h}$. An element of V^ω is said to have *weight* ω. The dimension of V^ω is called the *multiplicity* of ω in V. If $V^\omega \neq 0$, ω called a *weight* of V.

Proposition 2.22. *(a) One has $\mathfrak{g}^\alpha V^\omega \subset V^{\omega+\alpha}$ if $\omega \in \mathfrak{h}^*$, $\alpha \in R$.*

(b) The sum $V' := \sum_\omega V^\omega$ is direct and a $\mathfrak{g}$-submodule of V.

One says that some $v \in V$ is a *primitive element of weight* ω of it satisfies the conditions

(i) $v \neq 0$ has weight ω.

(ii) $X_\alpha v = 0 \forall \alpha \in R^+$ (equivalently, $\forall \alpha \in S$).

Given a primitive element of weight ω, it generates an indecomposable $\mathfrak{g}$-module in which ω is a weight of multiplicity 1.

2.7 Irreducible modules with a highest weight

The notion of a highest weight representation of $\mathfrak{g}$ is basic and is generalized to the representation theory of Kac-Moody algebras. We present two basic results here.

Theorem 2.23. *Let V be an irreducible $\mathfrak{g}$-module containing a primitive element v of weight ω. Then*

(a) v is the only primitive element of V up to scaling; its weight ω is called the 'highest weight' of V.

(b) The weights π of V have the form

$$\pi = \omega - \sum m_i \alpha_i \quad \textit{with } m_i \in \mathbf{N}. \tag{2.17}$$

They have finite multiplicity; in particular, ω has multiplicity 1. One has $V = \sum_\pi V^\pi$.

(c) For two irreducible $\mathfrak{g}$-modules V_1 and V_2 with highest weights ω_1 and ω_2 to be isomorphic, it is necessary and sufficient for $\omega_1 = \omega_2$.

Remark. *One can give examples of irreducible $\mathfrak{g}$-modules with no highest weight. These modules are necessarily infinite-dimensional.*

Theorem 2.24. *For each $\omega \in \mathfrak{h}^*$, there exists an irreducible $\mathfrak{g}$-module with highest weight equal to ω.*

Remark. *The above theorems establish a bijection between the elements of $\mathfrak{h}^*$ and the classes of irreducible $\mathfrak{g}$-modules with highest weight.*

2.8 Finite-dimensional modules

Proposition 2.25. *Let V be a finite-dimensional $\mathfrak{g}$-module. Then*

(a) $V = \sum_\pi V^\pi$.

(b) If π is a weight of V, $\pi(H_\alpha) \in \mathbf{Z} \forall \alpha \in R$.

(c) If $V \neq 0$, V contains a primitive element.

(d) If V is generated by a primitive element, then V is irreducible.

Theorem 2.26. *Let $\omega \in \mathfrak{h}^*$ and let E_ω be an irreducible $\mathfrak{g}$-module having ω as highest weight. For E_ω to be finite-dimensional, it is necessary and sufficient that*

$$\forall \alpha \in R^+, \omega(H_\alpha) \in \mathbf{Z}_{\geqslant 0}. \tag{2.18}$$

3 Kac-Moody algebras

Having summarized the basics of the structure theory of semisimple Lie algebras over $\mathbf{C}$ and the basics of their highest-weight linear representations, we may now proceed naturally to the formulation of their generalization. The key idea is to generalize the Cartan matrix of section 2.4.4; as the existence and uniqueness theorems show, every Cartan matrix encoding the data corresponding to a reduced root system determines a unique finite-dimensional semisimple complex Lie algebra. Therefore, generalizing the Cartan matrix provides a neat way to generalize the Lie algebra.

Note that the formulation of the generalized Cartan matrix is not the only way to approach Kac-Moody algebras. Early on, Kac considered arbitrary simple $\mathbf{Z}$-graded Lie algebras $\mathfrak{g} = \bigoplus_j \mathfrak{g}_j$ of 'finite growth,' meaning dim $\mathfrak{g}_j$ grew only polynomially in j. Nevertheless, we will present the approach of the generalized Cartan matrix here, following Kac [Kac94].

3.1 Basic definitions

We start with a complex $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ of rank ℓ and we will associate with it a complex Lie algebra $\mathfrak{g}(A)$.

Definition. *We call A a* generalized Cartan matrix *if it satisfies the following conditions:*

$$a_{ii} = 2 \text{ for } i = 1, \ldots, n \tag{C1}$$

$$a_{ij} \text{ are nonpositive integers for } i \neq j \tag{C2}$$

$$a_{ij} = 0 \text{ implies } a_{ji} = 0. \tag{C3}$$

However, we may start with an even more general object than a generalized Cartan matrix; let now A be *any* arbitrary matrix. A *realization* of A is a triple $\left(\mathfrak{h}, \Pi, \Pi^\vee\right)$, where $\mathfrak{h}$ is a complex vector space, $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \left\{\alpha_1^\vee, \ldots, \alpha_n^\vee\right\} \subset \mathfrak{h}$ are indexed subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, satisfying the conditions

$$\text{both sets } \Pi \text{ and } \Pi^\vee \text{ are linearly independent;} \tag{D1}$$

$$\left\langle \alpha_i^\vee, \alpha_j \right\rangle = a_{ij} \text{ for } i, j = 1, \ldots, n; \tag{D2}$$

$$n - \ell = \dim \mathfrak{h} - n. \tag{D3}$$

Two realizations $\left(\mathfrak{h}, \Pi, \Pi^\vee\right)$ and $\left(\mathfrak{h}', \Pi', \Pi'^\vee\right)$ are called *isomorphic* if there exists a vector space isomorphism $\phi : \mathfrak{h} \to \mathfrak{h}'$ such that $\phi\left(\Pi^\vee\right) = \Pi'^\vee$ and $\phi^*\left(\Pi'\right) = \Pi$.

Proposition 3.1. *Fix an $n \times n$ matrix A. Then there exists a realization of A, and this realization is unique up to isomorphism.*

Given two matrices A_1 and A_2 and their realizations $\left(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee\right)$ and $\left(\mathfrak{h}_2, \Pi_2, \Pi_2^\vee\right)$, we obtain a realization of the direct sum $A_1 \oplus A_2$ of the two matrices as

$$\left(\mathfrak{h}_1 \oplus \mathfrak{h}_2, \Pi_1 \times \{0\} \cup \{0\} \times \Pi_2, \Pi_1^\vee \times \{0,\} \cup \{0\} \times \Pi_2^\vee\right), \tag{3.1}$$

which establishes the notion of direct sum of realizations. Therefore, we can talk about decomposing realizations; a matrix A is called *decomposable* if, after reordering its indices (permuting the rows and columns by the same permutation), A decomposes into a nontrivial direct sum. Hence every realization decomposes into a direct sum of indecomposable realizations.

We now introduce some terminology that strengthens the analogy with the finite-dimensional case. We call Π the *root base*, $\Pi^\vee$ the *coroot base*, and elements from Π (resp. $\Pi^\vee$) *simple roots* (resp. *simple coroots*). We also put

$$Q := \sum_{i=1}^{n} \mathbf{Z}\alpha_i, \qquad Q_+ := \sum_{i=1}^{n} \mathbf{Z}_+\alpha_i, \tag{3.2}$$

and call the lattice Q the *root lattice*. Given any $\alpha = \sum_i k_i\alpha_i \in Q$, the number $\operatorname{ht}\alpha := \sum_i k_i$ is called the *height* of α. We put a partial order on $\mathfrak{h}^*$ by setting $\alpha \geqslant \mu$ if $\lambda - \mu \in Q_+$.

3.1.1 Construction of the auxiliary Lie algebra

Given an $n \times n$ complex matrix A and a realization $\left(\mathfrak{h}, \Pi, \Pi^\vee\right)$ of A, we first introduce an auxiliary Lie algebra $\tilde{\mathfrak{g}}(A)$ with the generators e_i, f_i ($i = 1, \ldots, n$) and $\mathfrak{h}$ and the following defining relations.

$$\begin{aligned} &[e_i, f_j] = \delta_{ij}\alpha_i^\vee && i, j = 1, \ldots, n, \\ &[h, h'] = 0 && h, h' \in \mathfrak{h}, \\ &[h, e_i] = \langle \alpha_i, h\rangle\, e_i && \\ &[h, f_i] = -\langle \alpha_i, h\rangle\, f_i && i = 1, \ldots, n; h \in \mathfrak{h}. \end{aligned} \tag{DR}$$

The uniqueness of the realization guarantees that $\tilde{\mathfrak{g}}(A)$ depends only on A. Denote by $\tilde{\mathfrak{n}}_+$ (resp $\tilde{\mathfrak{n}}_-$) the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by $e_1, \ldots, e_n$ (resp. $f_1, \ldots, f_n$). We now record the following properties of the auxiliary Lie algebra $\tilde{\mathfrak{g}}(A)$.

Theorem 3.2. *(a)* $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$ *as vector spaces.*

(b) $\tilde{\mathfrak{n}}_+$ *(resp.* $\tilde{\mathfrak{n}}_-$*) is freely generated by* $e_1, \ldots, e_n$ *(resp.* $f_1, \ldots, f_n$*).*

(c) The map $e_i \mapsto -f_i$, $f_i \mapsto -e_i$ ($i = 1, \ldots, n$), $h \mapsto -h$ ($h \in \mathfrak{h}$) *can be uniquely extended to an involution of the Lie algebra* $\tilde{\mathfrak{g}}(A)$.

(d) With respect to $\mathfrak{h}$ *one has the root space decomposition*

$$\tilde{\mathfrak{g}}(A) = \Big(\bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \tilde{\mathfrak{g}}_{-\alpha}\Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\substack{\alpha \in Q \\ \alpha \neq 0}} \tilde{\mathfrak{g}}_{\alpha}\Big), \tag{3.3}$$

where $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}}(A) : [h, x] = \alpha(h)x\}$. *Additionally,* $\dim \tilde{\mathfrak{g}}_\alpha < \infty$, *and* $\tilde{\mathfrak{g}}_\alpha \subset \tilde{\mathfrak{n}}_\pm$ *for* $\pm\alpha \in Q_+, \alpha \neq 0$.

(e) Among the ideals of $\tilde{\mathfrak{g}}(A)$ *intersecting* $\mathfrak{h}$ *trivially, there exists a unique maximal ideal* $\mathfrak{r}$. *Furthermore, there is a direct sum decomposition of ideals*

$$\mathfrak{r} = (\mathfrak{r} \cap \tilde{\mathfrak{n}}_-) \oplus (\mathfrak{r} \cap \tilde{\mathfrak{n}}_+)\,. \tag{3.4}$$

The analogy to the root space of a finite-dimensional semisimple Lie algebra is clear.

3.1.2 Construction of the Kac-Moody algebra

We now may proceed to construct the Lie algebra $\mathfrak{g}(A)$ that is the main object of study. With $(\mathfrak{h}, \Pi, \Pi^\vee)$ as before and $\tilde{\mathfrak{g}}(A)$ the auxiliary Lie algebra associated to this realization on generators e_i, f_i ($i = 1, \ldots, n$), and $\mathfrak{h}$, and the defininining relations (DR). By Theorem 3.2, the natural map $\mathfrak{h} \to \tilde{\mathfrak{g}}(A)$ is an embedding. Let $\mathfrak{r}$ be the maximal ideal in $\tilde{\mathfrak{g}}(A)$ intersecting $\mathfrak{h}$ trivially, as in Theorem 3.2(e). Then, put

$$\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\mathfrak{r}. \tag{3.5}$$

The matrix A is called the *Cartan matrix* of the Lie algebra $\mathfrak{g}(A)$, and n is called the *rank* of $\mathfrak{g}(A)$. We say $(\mathfrak{g}(A), \mathfrak{h}, \Pi, \Pi^\vee)$ is the *quadruple* associated to A, and say two quadruples $(\mathfrak{g}(A), \mathfrak{h}, \Pi, \Pi^\vee)$ and $(\mathfrak{g}(A'), \mathfrak{h}', \Pi', \Pi'^\vee)$ are *isomorphic* if there exists a Lie algebra isomorphism $\phi : \mathfrak{g}(A) \to \mathfrak{g}(A')$ such that $\phi(\mathfrak{h}) = \mathfrak{h}'$, $\phi(\Pi^\vee) = \Pi'^\vee$, and $\phi^*(\Pi') = \Pi$. Recall the conditions (C1–C3) on a generalized Cartan matrix. We are in a position to make the following

Definition. *The Lie algebra $\mathfrak{g}(A)$ whose Cartan matrix is a generalized Cartan matrix is called a* Kac-Moody algebra.

We re-use the notation $e_i, f_i, \mathfrak{h}$ for the images of these generators in $\mathfrak{g}(A)$ after the quotient by $\mathfrak{r}$. In analogy with the classical finite-dimensional case, we call the subalgebra $\mathfrak{h} \subset \mathfrak{g}(A)$ the *Cartan subalgebra* and the elements e_i, f_i the *Chevalley generators* of $\mathfrak{g}(A)$. The Chevalley generators generate the derived subalgebra $\mathfrak{g}'(A) := D^2\mathfrak{g}(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]$, and

$$\mathfrak{g}(A) = \mathfrak{g}'(A) + \mathfrak{h}, \tag{3.6}$$

where this decomposition is not necessarily direct.

Remark. $\mathfrak{g}(A) = \mathfrak{g}'(A)$ *if and only if* $\det A \neq 0$.

Quadruples decompose like realizations.

Proposition 3.3. *The quadruple associated to a direct sum of matrices A_i is isomorphic to a direct sum of the quadruples associated to the A_i. The root system of $\mathfrak{g}(A)$ is the union of the root systems of the $\mathfrak{g}(A_i)$.*

We now characterize the center of the Lie algebra $\mathfrak{g}(A)$.

Proposition 3.4. *The center of the Lie algebra $\mathfrak{g}(A)$ is equal to*

$$\mathfrak{c} := \{h \in \mathfrak{h} : \langle \alpha_i, h \rangle = 0 \forall i = 1, \ldots, n\}. \tag{3.7}$$

Furtherore, $\dim \mathfrak{c} = n - \ell$, *where* $\ell = \operatorname{rk} A$.

We record the necessary and sufficient conditions on the matrix A for the algebra $\mathfrak{g}(A)$ to be simple.

Proposition 3.5. $\mathfrak{g}(A)$ *is simple if and only if* $\det A \neq 0$ *and for each pair of indices i and j, there exist indices $i_1, i_2, \ldots, i_s$ such that the product $a_{ii_1} a_{i_1 i_2} \cdots a_{i_s j} \neq 0$.*

3.1.3 Root space of the Kac-Moody algebra $\mathfrak{g}(A)$

Continuing in analogy, put $\mathfrak{h}' := \sum_{i=1}^n \mathbf{C}\alpha_i^\vee$. Then $\mathfrak{g}'(A) \cap \mathfrak{h} = \mathfrak{h}'$, and $\mathfrak{g}'(A) \cap \mathfrak{g}_\alpha = \mathfrak{g}_\alpha$ if $\alpha \neq 0$. One may deduce from Theorem 3.2(d) the following *root space decomposition* with respect to $\mathfrak{h}$.

$$\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha. \tag{3.8}$$

In further analogy with section 2.5.2, we have $\mathfrak{g}_0 = \mathfrak{h}$, and we define the *multiplicity* $\operatorname{mult}\alpha := \dim \mathfrak{g}_\alpha$. Note that $\operatorname{mult}\alpha \leqslant n^{|\operatorname{ht}\alpha|}$.

Call a nonzero element $\alpha \in Q$ a *root* if $\operatorname{mult}\alpha \neq 0$. A root $\alpha > 0$ (resp. < 0) is called *positive* (resp. *negative*), and (3.3) assures us that α must be either positive or negative. Denoting the set of roots (resp. positive, negative roots) by Δ (resp. Δ_+, Δ_-), Δ decomposes disjointly as

$$\Delta = \Delta_+ \sqcup \Delta_-. \tag{3.9}$$

Let $\mathfrak{n}_+$ (resp $\mathfrak{n}_-$) denote the subalgebra of $\mathfrak{g}(A)$ generated by $e_1, \dots, e_n$ (resp. $f_1, \dots, f_n$). Theorem 3.2(a) establishes the *triangular decomposition* of vector spaces

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+. \tag{3.10}$$

It follows from Theorem 3.2(e) that $\mathfrak{r} \subset \tilde{\mathfrak{g}}(A)$ is $\tilde{\omega}$-invariant. Filtering through the quotient, $\tilde{\omega}$ induces an involution ω of $\mathfrak{g}(A)$ called the *Chevalley involution* of $\mathfrak{g}(A)$. As $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, we have that $\operatorname{mult}\alpha = \operatorname{mult}(-\alpha)$ and $\Delta_- = -\Delta_+$.

3.2 Invariant bilinear form

Though we omitted it in our discussion, every finite-dimensional Lie algebra carries a distinguished symmetric bilinear form called the Killing form. We now present some results pertaining to the existence of invariant bilinear forms on Kac-Moody algebras.

Such a form only exists on *symmetrizable* Kac-Moody algebras. We first explain what this terminology means. On the level of matrices, an $n \times n$ matrix A is called *symmetrizable* if there exists an invertible diagonal matrix D with $d_{ii} = \epsilon_i$, and a symmetric matrix B such that $A = DB$. The matrix B is called a *symmetrization* of A and $\mathfrak{g}(A)$ is called a *symmetrizable* Lie algebra, since rescaling the Chevalley generators $e_i \mapsto e_i$, $f_i \mapsto \epsilon_i f_i$ for $\epsilon_i \neq 0$ rescales $\alpha_i^\vee \mapsto \epsilon_i \alpha_i^\vee$, which extends to an isomorphism $\mathfrak{h} \mapsto \mathfrak{h}$ (nonunique, if $\det A = 0$), which in turn extends to an isomorphism of Lie algebras $\mathfrak{g}(A) \to \mathfrak{g}(DA)$.

Let A be a symmetrizable matrix with a fixed decomposition $A = DB$ as above, and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A. Fix a complementary subspace $\mathfrak{h}''$ to $\mathfrak{h}' = \sum \mathbf{C}\alpha_i^\vee$ in $\mathfrak{h}$, and define a symmetric bilinear $\mathbf{C}$-valued form $(\, , \,)$ on $\mathfrak{h}$ by the two equations

$$(\alpha_i^\vee, h) = \langle \alpha_i, h \rangle \epsilon_i \text{ for } h \in \mathfrak{h}, i = 1, \dots, n \tag{3.11}$$

$$(h', h'') = 0 \text{ for } h', h'' \in \mathfrak{h}''. \tag{3.12}$$

By the decomposition $A = DB$ and since $\alpha_1^\vee, \dots, \alpha_n^\vee$ are linearly independent, we conclude from (3.11) that

$$(\alpha_i^\vee, \alpha_j^\vee) = b_{ij}\epsilon_i\epsilon_j \text{ for } i = 1, \dots, n, \tag{3.13}$$

so there is no ambiguity in the definition of $(\, , \,)$.

Proposition 3.6. *The bilinear form $(\, , \,)$ is nondegenerate on $\mathfrak{h}$.*

Since the bilinear form $(\, , \,)|_{\mathfrak{h}}$ is nondegenerate, we have an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ given by

$$\langle \nu(h), h_1 \rangle = (h, h_1), \quad h, h_1 \in \mathfrak{h}, \tag{3.14}$$

and the induced bilinear form $(\, , \,)$ on $\mathfrak{h}^*$. Then (3.11) implies

$$\nu(\alpha_i^\vee) = \epsilon_i \alpha_i, \quad i = 1, \dots, n. \tag{3.15}$$

Combining with (3.13) yields

$$(\alpha_i, \alpha_j) = b_{ij} = a_{ij}/\epsilon_i, \quad i, j = 1, \dots, n. \tag{3.16}$$

Theorem 3.7. *Lt $\mathfrak{g}(A)$ be a symmetrizable Lie algebra; fix a decomposition of A. Then there exists a nondegenerate symmetric bilinear complex-valued form $(\ ,\ )$ on $\mathfrak{g}(A)$ such that*

(a) $(\ ,\ )$ *is invariant; i.e.* $([x, y], z) = (x, [y, z]) \forall x, y \in \mathfrak{g}(A)$.

(b) $(\ ,\ )|_{\mathfrak{h}}$ *is defined by* (3.11) *and* (3.12) *and is nondegenerate.*

(c) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *if* $\alpha + \beta \neq 0$.

(d) $(\ ,\ )|_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}}$ *is nondegenerate for* $\alpha \neq 0$*, and hence* $\mathfrak{g}_\alpha$ *and* $\mathfrak{g}_{-\alpha}$ *are nondegenerately paired by* $(\ ,\ )$.

(e) $[x, y] = (x, y)\nu^{-1}(\alpha)$ *for* $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}, \alpha \in \Delta$.

If, moreover, $(\alpha_i, \alpha_i) > 0$ for $i = 1, \ldots, n$, $(\ ,\ )$ is called a *standard* invariant form.

3.3 Casimir element

Let $\mathfrak{g}(A)$ be a Lie algebra associated to a matrix A, $\mathfrak{h}$ the Cartan subalgebra, and $\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$ the root space decomposition with respect to $\mathfrak{h}$.

Definition. *A $\mathfrak{g}(A)$-module V is called* restricted *if for every $v \in V$, we have $\mathfrak{g}_\alpha(v) = 0$ for all but a finite number of positive roots α.*

Remark. *Every submodule or quotient of a restricted module is restricted. Direct sums and tensor products of a finite number of restricted modules are restricted.*

Assume now that A is symmetrizable and $(\ ,\ )$ an invariant bilinear form (guaranteed by Theorem 3.7). Given a restricted $\mathfrak{g}(A)$-module V, we construct a linear operator Ω on V, called the (generalized) *Casimir operator*, as follows.

Define implicity a linear function $\rho \in \mathfrak{h}^*$ by

$$\langle \rho, \alpha_i^\vee \rangle = \frac{1}{2} a_{ii} \quad i = 1, \ldots, n. \tag{3.17}$$

If $\det A = 0$, this does not specify ρ uniquely, and hence we must choose one. Then (3.15) and (3.16) imply

$$(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i) \quad i = 1, \ldots, n. \tag{3.18}$$

Next, for each $\alpha \in \Delta_+$, choose a basis $\left\{e_\alpha^{(i)}\right\}$ of the space $\mathfrak{g}_\alpha$, and form the dual basis $\left\{e_{-\alpha}^{(i)}\right\}$ of $\mathfrak{g}_{-\alpha}$. We then define an operator Ω_0 on V by

$$\Omega_0 := 2 \sum_{\alpha \in \Delta_+} \sum_i e_{-\alpha}^{(i)} e_\alpha^{(i)}. \tag{3.19}$$

Ω_0 is independent of the choice of basis. Since $\forall v \in V$, only a finite number of summands are nonzero, Ω_0 is well-defined on V. Let $u_1, u_2, \ldots$ and $u^1, u^2, \ldots$ be dual bases of $\mathfrak{h}$. The generalized Casimir operator is defined by

$$\Omega = 2\nu^{-1}(\rho) + \sum_i u^i u_i + \Omega_0. \tag{3.20}$$

We now record the following theorem that justifies this construction.

Theorem 3.8. *Let $\mathfrak{g}(A)$ be a symmetrizable Lie algebra and V a restricted $\mathfrak{g}(A)$-module. Then the Casimir operator Ω commutes with the action of $\mathfrak{g}(A)$ on V.*

We now give a result about eigenvalues of Ω.

Corollary 3.9. *Under the same hypotheses as Theorem 3.8, if there exists some $v \in V$ such that the Chevalley generators e_i vanish on v for $i = 1, \ldots, n$, and $h(v) = \langle \Lambda, h \rangle v$ for some $\Lambda \in \mathfrak{h}^*$ and all $h \in \mathfrak{h}$, then*

$$\Omega(v) = (\Lambda + 2\rho, \Lambda)v. \tag{3.21}$$

3.4 Integrable representations

We first introduce some definitions in close analogy with the classical case. A $\mathfrak{g}(A)$-module V is called $\mathfrak{h}$-*diagonalizable* if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where

$$V_\lambda = \{v \in V : h(v) = \langle \lambda, h \rangle v \forall h \in \mathfrak{h}\}. \tag{3.22}$$

V_λ is called a *weight space*, $\lambda \in \mathfrak{h}^*$ is called a *weight* if $V_\lambda \neq 0$, and $\dim V_\lambda$ is called the *multiplicity* of λ and is denoted $\operatorname{mult}_V \lambda$. A $\mathfrak{h}$-diagonalizable module over a Kac-Moody algebra $\mathfrak{g}(A)$ is called *integrable* if all e_i and f_i are locally nilpotent on V. (An operator T is locally nilpotent on a vector space V if T is nilpotent on every finite dimensional subspace of V.)

Remark. *$\mathfrak{g}(A)$ is an integrable $\mathfrak{g}(A)$-module under the adjoint representation; i.e.* $\operatorname{ad} e_i$ *and* $\operatorname{ad} f_i$ *are locally nilpotent on $\mathfrak{g}(A)$.*

The following two propositions together justify the terminology 'integrable.'

Proposition 3.10. *Let $\mathfrak{g}(A)$ be a Kac-Moody algebra and let e_i, f_i ($i = 1, \ldots, n$) be its Chevalley generators. Put $\mathfrak{g}_{(i)} = \mathbf{C}e_i + \mathbf{C}\alpha_i^\vee + \mathbf{C}f_i$. Then*

$$\mathfrak{g}_{(i)} \simeq \mathfrak{sl}_2(\mathbf{C}) \tag{3.23}$$

as $\mathbf{C}$-algebras with standard basis $\left\{e_i, \alpha_i^\vee, f_i\right\}$.

Proposition 3.11. *Let V be an integrable $\mathfrak{g}(A)$-module. As a $\mathfrak{g}_{(i)}$-module, V decomposes into a direct sum of finite-dimensional irreducible $\mathfrak{h}$-invariant modules. (Hence the action of $\mathfrak{g}_{(i)}$ can be 'integrated' to the action of the group $SL_2(\mathbf{C})$.)*

3.5 Weyl group

Definition. *For each $i = 1, \ldots, n$, we define the* fundamental reflection r_i *of the space $\mathfrak{h}^*$ by*

$$r_i(\lambda) = \lambda - \left\langle \lambda, \alpha_i^\vee \right\rangle \alpha_i, \alpha \in \mathfrak{h}^*. \tag{3.24}$$

One can verify r_i is a reflection since its fixed point set is $T_i = \left\{\lambda \in \mathfrak{h}^* : \left\langle \lambda, \alpha_i^\vee \right\rangle = 0\right\}$, and $r_i(\alpha_i) = -\alpha_i$.

Definition. *The subgroup $W \subset GL(\mathfrak{h}^*)$ generated by all fundamental reflections is called the* Weyl group *W of $\mathfrak{g}(A)$.*

The following proposition justifies this definition.

Proposition 3.12. *Let V be an integrable module over a Kac-Moody algebra $\mathfrak{g}(A)$. Then*

(a) $\operatorname{mult}_V w(\lambda) \forall \lambda \in \mathfrak{h}^*, w \in W$. *In particular, the set of weights of V is W-invariant.*

(b) The root system Δ of $\mathfrak{g}(A)$ is W-invariant, and $\operatorname{mult} \alpha = \operatorname{mult} w(\alpha) \forall \alpha \in \Delta, w \in W$.

We now consider the special case of a symmetrizable Kac-Moody algebra.

Proposition 3.13. *Let A be a symmetrizable generalized Cartan matrix and let $(\ , \ )$ be a standard invariant bilinear form on $\mathfrak{g}(A)$. The restriction of the bilinear form $(\ , \ )$ to $\mathfrak{h}^*$ is W-invariant.*

3.5.1 Weyl chambers

It is basic to understand the geometry of the action of the Weyl group. First, let A be an $n \times n$ matrix over $\mathbf{R}$, and let $\left(\mathfrak{h}_{\mathbf{R}}, \Pi, \Pi^{\vee}\right)$ be a realization of the matrix A over $\mathbf{R}$; i.e. $\mathfrak{h}_{\mathbf{R}}$ is a real vector space of dimension $2n - \ell$, so that, extending $\mathfrak{h}_{\mathbf{R}}$ by scalars to $\mathfrak{h} := \mathfrak{h}_{\mathbf{R}} \otimes \mathbf{C}$, $\left(\mathfrak{h}, \Pi, \Pi^{\vee}\right)$ is a realization of A over $\mathbf{C}$.

Remark. *$\mathfrak{h}_{\mathbf{R}}$ is stable under W since $Q^{\vee} \subset \mathfrak{h}_{\mathbf{R}}$, where Q is the coroot lattice (c.f. (3.2) for the definition of the root lattice).*

Definition. *The set*

$$C := \{h \in \mathfrak{h}_{\mathbf{R}} : \langle \alpha_i, h \rangle \geqslant 0 \text{ for } i = 1, \dots, n\} \tag{3.25}$$

is called the fundamental chamber.

Definition. *The sets $w(C)$, $w \in W$ are called* chambers, *and their union*

$$X = \bigcup_{w \in W} w(C) \tag{3.26}$$

is called the Tits cone.

Remark. *We have the corresponding dual notions of $C^{\vee}$ and $X^{\vee}$ in $\mathfrak{h}_{\mathbf{R}}^*$.*

We now record a proposition that characterizes the action of the Weyl group on chambers.

Proposition 3.14. *(a) For $h \in C$, the group $W_h = \{w \in W : w(h) = h\}$ is generated by the fundaental reflections that it contains.*

(b) The fundamental chamber C is a fundamental domain for the action of W on X; i.e. any orbit $W \cdot h$ of $h \in X$ intersects C in exactly one point. In particular, W acts simply transitively on chambers.

(c) $X = \{x \in \mathfrak{h}_{\mathbf{R}} : \langle \alpha, h \rangle < 0$ only for a finite number of $\alpha \in \Delta_+\}$. In particular, X is a convex cone.

(d) $C = \left\{h \in \mathfrak{h}_{\mathbf{R}} : \forall w \in W, h - w(h) = \sum_i c_i \alpha_i^{\vee},\right.$ where $\left. c_i \geqslant 0\right\}$.

(e) The following conditions are equivalent:

(i) $|W| < \infty$;

(ii) $X = \mathfrak{h}_{\mathbf{R}}$;

(iii) $|\Delta| < \infty$;

(iv) $\left|\Delta^{\vee}\right| < \infty$.

(f) If $h \in X$, then $|W_h| < \infty$ if and only if h lies in the interior of X.

3.6 Classification of generalized Cartan matrices

It is a convenient fact about Kac-Moody algebras that basic properties of the generalized Cartan matrix A translate into essential facts about the algebra $\mathfrak{g}(A)$. Suppose A is a real $n \times n$ matrix satisfying the following three properties (we may reduce to the first and a generalized Cartan matrix satisfies the rest).

(m1) A is indecomposable.

(m2) $a_{ij} \leqslant 0$ for $i \neq j$.

(m3) $a_{ij} = 0$ implies $a_{ji} = 0$.

The main result is then as follows.

Theorem 3.15. *Let A be a real $n \times n$ matrix satisfying (m1), (m2), and (m3). Then one and only one of the following three possibilities holds.*

(Fin) $\det A \neq 0$; $\exists u > 0 : Au > 0$; $Av \geqslant 0$ *implies* $v > 0$ *or* $v = 0$.

(Aff) $\operatorname{corank} A = 1$; $\exists u > 0 : Au = 0$; $Av \geqslant 0$ *implies* $Av = 0$;

(Ind) $\exists u > 0 : Au < 0$; $Av \geqslant 0, v \geqslant 0$ *imply* $v = 0$.

Definition. *We say that a matrix satisfying (Fin), (Aff), and (Ind) is of* finite, affine, *or* indefinite *type, respectively.*

Corollary 3.16. *Let A be a matrix satisfying (m1), (m2), and (m3). Then A is of finite, affine, or indefinite type if and only if there exists an $\alpha > 0$ such that $A\alpha > 0, = 0$, or < 0, respectively.*

3.6.1 Kac-Moody algebras of finite type

One success of the generalization of Kac-Moody is that one recovers the full collection of classical finite-dimensional complex semisimple Lie algebras in the set of Kac-Moody algebras of finite type.

Proposition 3.17. *Let A be an indecomposable generalized Cartan matrix. Then the following conditions are equivalent.*

(i) *A is a generalized Cartan matrix of finite type;*

(ii) *A is symmetrizable and the bilinear form $(\ , \ )_{\mathfrak{h}_\mathbb{R}}$ is positive-definite;*

(iii) $|W| < \infty$;

(iv) $|\Delta| < \infty$;

(v) *$\mathfrak{g}(A)$ is a simple finite-dimensional Lie algebra;*

(vi) $\exists \alpha \in \Delta_+ : \alpha + \alpha_i \notin \Delta \forall i = 1, \dots, n$.

3.7 Root system

The main innovation in the theory of roots of a Kac-Moody algebra $\mathfrak{g}(A)$ vis-à-vis the classical theory, is the notion of an imaginary root, which is an entirely new notion. The classical theory of roots is recovered in the complementary notion of a real root.

3.7.1 Real roots

Definition. *A root $\alpha \in \Delta$ is called* real *if there exists a $w \in W$ such that $w(\alpha)$ is a simple root.*

We denote the set of all real roots by Δ^{re} and the set of all positive real roots by Δ^{re}_+. Given any $\alpha \in \Delta^{\text{re}}$, then $\alpha = w(\alpha_i)$ for some $\alpha_i \in \Pi$, $w \in W$. Define the *dual* (real) root $\alpha^\vee \in \Delta^{\vee\text{re}}$ by $\alpha^\vee = w\left(\alpha_i^\vee\right)$. This is independent of the choice of presentation $\alpha = w\left(\alpha_i\right)$. We define a reflection r_α with respect to $\alpha \in \Delta^{\text{re}}$ by

$$r_\alpha(\lambda) = \lambda - \left\langle \lambda, \alpha^\vee \right\rangle \alpha, \quad \lambda \in \mathfrak{h}^*. \tag{3.27}$$

The following proposition strongly evokes the classical setting.

Proposition 3.18. *Let α be a real root of a Kac-Moody algebra $\mathfrak{g}(A)$. Then*

(a) $\operatorname{mult} \alpha = 1$.

(b) *$k\alpha$ is a root if and only if $k = \pm 1$.*

(c) *If $\beta \in \Delta$, then $\exists p, q \in \mathbf{Z}_{\geqslant 0}$ related by the equation*

$$p - 1 = \left\langle \beta, \alpha^\vee \right\rangle, \tag{3.28}$$

such that $\beta + k\alpha \in \Delta \cup \{0\}$ if and only if $-p \leqslant k \leqslant q, k \in \mathbf{Z}$.

(d) *Suppose that A is symmetrizable and let $(\ ,\ )$ be a standard invariant bilinear form on $\mathfrak{g}(A)$. Then*

  (i) $(\alpha, \alpha) > 0$,

  (ii) $\alpha^\vee = 2\nu^{-1}(\alpha)/(\alpha, \alpha)$, *and*

  (iii) *if $\alpha = \sum_i k_i \alpha_i$, then $k_i(\alpha_i, \alpha_i) \in (\alpha, \alpha)\mathbf{Z}$.*

(e) *Provided that $\pm\alpha \notin \Pi$, $\exists i$ such that*

$$|\operatorname{ht} r_i(\alpha)| < |\operatorname{ht} \alpha| . \tag{3.29}$$

Definition. *Let A be a symmetrizable generalized Cartan matrix and $(\ ,\ )$ a standard invariant bilinear form. Then, given a real root α, we have $(\alpha, \alpha) = (\alpha_i, \alpha_i)$ for some simple root α_i. We call α a* short *(resp.* long*) real root if $(\alpha, \alpha) = \min_i(\alpha_i, \alpha_i)$ (resp. $(\alpha, \alpha) = \max_i(\alpha_i, \alpha_i)$).*

Remark. *The definitions of short and long roots are independent of the choice of a standard form. If A is symmetric, then all simple roots and hence all real roots have the same square length.*

3.7.2 Imaginary roots

Definition. *A root $\alpha \in \Delta$ is called* imaginary *if it is not real.*

We denote the set of all imaginary roots by Δ^{im} and the set of all positive imaginary roots by Δ^{im}_+. By definition, Δ decomposes as a disjoint union like

$$\Delta = \Delta^{\text{re}} \sqcup \Delta^{\text{im}}. \tag{3.30}$$

Imaginary roots enjoy the following properties, some very different from those of real roots.

Proposition 3.19. (a) *The set Δ^{im}_+ is W-invariant.*

(b) *For $\alpha \in \Delta^{\text{im}}_+$, $\exists! \beta \in -C^\vee$ (i.e. $\left\langle \beta, \alpha_i^\vee \right\rangle \leqslant 0 \forall i$) which is W-equivalent to α.*

(c) *If A is symmetrizable and $(\ ,\ )$ is a standard invariant bilinear form, then $\alpha \in \Delta$ is imaginary if and only if $(\alpha, \alpha) \leqslant 0$.*

(d) *If $\alpha \in \Delta^{\text{im}}_+$ and r is a nonzero rational number such that $r\alpha \in Q$, then $r\alpha \in \Delta^{\text{im}}$. In particular, $n\alpha \in \Delta^{\text{im}}$ if $n \in \mathbf{Z} - \{0\}$.*

We now state when imaginary roots exist.

Theorem 3.20. *Let A be an indecomposable generalized Cartan matrix.*

(a) *If A is of finite type, $\Delta^{\text{im}} = \emptyset$.*

(b) *If A is of affine type, then*

$$\Delta_+^{\text{im}} = \{n\delta, n \in \mathbf{Z}\}, \tag{3.31}$$

where $\delta = \sum_{i=0}^{\ell} a_i\alpha_i$ is a linear combination of roots α_i in terms of constants a_i, which are effectively computable from the matrix A,[1] *and*

(c) *If A is of indefinite type, then $\exists\alpha \in \Delta_+^{\text{im}}, \alpha = \sum_i k_i\alpha_i$, such that $k_i > 0$ and $\langle\alpha, \alpha_i^\vee\rangle < 0 \forall i = 1, \ldots, n$.*

The Tits cone X may be described naturally in terms of imaginary roots.

Proposition 3.21. (a) *If A is of finite type, then $X = \mathfrak{h}_{\mathbf{R}}$.*

(b) *If A is of affine type, then*

$$X = \{h \in \mathfrak{h}_{\mathbf{R}} : \langle\delta, h\rangle > 0\} \cup \mathbf{R}\nu^{-1}(\delta). \tag{3.32}$$

(c) *If A is of indefinite type, then*

$$\overline{X} = \left\{h \in \mathfrak{h}_{\mathbf{R}} : \langle\alpha, h\rangle \geqslant 0 \text{ for all } \alpha \in \Delta_+^{\text{im}}\right\}, \tag{3.33}$$

where $\overline{X}$ denotes the closure of X in the metric topology of $\mathfrak{h}_{\mathbf{R}}$.

Remark. *Proposition 3.21 has a geometric interpretation in terms of the* imaginary cone Z, *the convex hull in $\mathfrak{h}_{\mathbf{R}}^*$ of the set $\Delta_+^{\text{im}} \cup \{0\}$. Namely, th cones $\overline{Z}$ and $\overline{X}$ are dual to each other;*

$$\overline{X} = \left\{h \in \mathfrak{h}_{\mathrm{R}} : \langle\alpha, h\rangle \geqslant 0 \forall\alpha \in \overline{Z}\right\}. \tag{3.34}$$

In particular, Z is a convex cone.

We introduce the notion of a Kac-Moody root base and consider the action of the Weyl group on it.

Definition. *A linearly independent set of roots $\mathrm{II}' = \{\alpha_1', \alpha_2', \ldots\}$ is called a* root base *of Δ if each root α can be written in the form $\alpha = \pm\sum_i k_i\alpha_i'$, where $k_i \in \mathbf{Z}_+$.*

Proposition 3.22. *Let A be an indecoposable generalized Cartan matrix. Then any root base II' of Δ is W-conjugate to II or $-\mathrm{II}$.*

Remark. II *is W-conjugate to $-\mathrm{II}$ if and only if A is of finite type.*

3.7.3 Hyperbolic type

Definition. *A generalized Cartan matrix A is called a matrix of* hyperbolic *type if it is indecomposable symmetrizable of indefinite type, and if every proper connected subdiagram of the Dynkin diagram associated to A is of finite or affine type.*

Proposition 3.23. *Let A be a generalized Cartan matrix of finite, affine, or hyperbolic type. Then*

(a) *The set of all short real roots is*

$$\left\{\alpha \in Q : |\alpha|^2 = a = \min_i |\alpha_i|^2\right\}. \tag{3.35}$$

[1] The coefficients a_i are the labels of the Dynkin diagram associated to A, which we omit in our discussion, but which is treated in [Kac94, Ch. 4].

(b) *The set of all real roots is*

$$\left\{\alpha = \sum_j k_j\alpha_j \in Q : |\alpha|^2 > 0 \textit{ and } k_j|\alpha_j|^2 \in \mathbf{Z}\forall j\right\}. \tag{3.36}$$

(c) *The set of all imaginary roots is*

$$\left\{\alpha \in \mathbf{Q} - \{0\} : |\alpha|^2 \leqslant 0\right\}. \tag{3.37}$$

(d) *If A is affine, then there exist roots of intermediate squared length m if and only if $A = A_{2\ell}^{(2)}$, $\ell \geqslant 2$.*[2] *The set of such roots coincides with $\left\{\alpha \in Q : |\alpha|^2 = m\right\}$.*

3.8 Kac-Moody algebras of affine type

A convenient feature of Kac-Moody algebras of affine type is that all the basic information about them can be expressed in terms of corresponding objects for an 'underlying' simple *finite-dimensional* Lie algebra. We make this notion precise in what follows. Let A be a generalized Cartan matrix of affine type of order $\ell+1$ (and rank ℓ). Let $\mathfrak{g} = \mathfrak{g}(A)$ be the affine algebra associated to a matrix A of affine type; let $\mathfrak{h}$ be its Cartan subalgebra, $\Pi = \{\alpha_0, \ldots, \alpha_\ell\} \subset \mathfrak{h}^*$ the set of simple roots, $\Pi^\vee = \left\{\alpha_0^\vee, \ldots, \alpha_\ell^\vee\right\} \subset \mathfrak{h}$ the set of simple coroots, Δ the root system, and Q and $Q^\vee$ the root and coroot lattices. Proposition 3.4 implies that the center of $\mathfrak{g}$ is one-dimensional and spanned by a linear combination of coroots[3] called the *canonical central element* and denoted by K. More precisely, with a_i as in Theorem 3.20(b), let $a_i^\vee$ be the corresponding labels corresponding to the dual algebra. Then

$$K = \sum_{i=0}^{\ell} a_i^\vee \alpha_i^\vee. \tag{3.38}$$

Note that $[\mathfrak{g}, \mathfrak{g}]$ has corank 1 in $\mathfrak{g}$. Fixing an element $d \in \mathfrak{h}$ satisfying $\langle\alpha_i, d\rangle = 0 \forall i = 1, \ldots, n$ and $\langle\alpha_0, d\rangle = 1$, which we call the *scaling element*, it is clear that the elements $\alpha_0^\vee, \ldots, \alpha_\ell^\vee, d$ form a basis of $\mathfrak{h}$, and

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathbf{C}d. \tag{3.39}$$

There is a distinguished standard form on $\mathfrak{g}$ satisfying the properties of Theorem 3.7, called the *normalized invariant form*.

3.8.1 The underlying finite-dimensional Lie algebra

Denote by $\mathring{\mathfrak{h}}$ (resp. $\mathring{\mathfrak{h}}_{\mathbf{R}}$) the linear span over $\mathbf{C}$ (resp. $\mathbf{R}$) of the coroots $\alpha_1^\vee, \ldots, \alpha_\ell^\vee$. The dual notions $\mathring{\mathfrak{h}}^*$ and $\mathring{\mathfrak{h}}^*_{\mathbf{R}}$ are defined similarly. Then we have an orthogonal direct sum of subspaces

$$\mathfrak{h} = \mathring{\mathfrak{h}} \oplus (\mathbf{C}K + \mathbf{C}d) \qquad\qquad \mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus (\mathbf{C}\delta + \mathbf{C}\Lambda_0), \tag{3.40}$$

where δ is as in Theorem 3.20(b), and $\Lambda_0 \in \mathfrak{h}^*$ is defined implicitly by the equations

$$\left\langle\Lambda_0, \alpha_i^\vee\right\rangle = \delta_{0i} \text{ for } i = 0, \ldots, \ell; \qquad\qquad \langle\Lambda_0, d\rangle = 0. \tag{3.41}$$

Denote now by $\mathring{\mathfrak{g}}$ the subalgebra of $\mathfrak{g}$ generated by the e_i and f_i with $i = 1, \ldots, \ell$. $\mathring{\mathfrak{g}}$ is a Kac-Moody algebra associated to the matrix $\mathring{A}$ obtained from A by deleting the 0th row and column.

[2] This 'type' belongs to the classification of generalized Cartan matrices of affine type, and is given in terms of its Dynkin diagram in [Kac94, p.55].

[3] The coefficients in the linear combination again depend on the combinatorics of the Dynkin diagram.

The elements e_i, f_i ($i = 1, \ldots, \ell$) are the Chevalley generators of $\mathring{\mathfrak{g}}$, and $\mathring{\mathfrak{h}} = \mathring{\mathfrak{g}} \cap \mathfrak{h}$ is its Cartan subalgebra. $\mathring{\Pi} = \{\alpha_1, \ldots, \alpha_\ell\}$ is the root basis and $\mathring{\Pi}^\vee = \left\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\right\}$ the coroot basis for $\mathring{\mathfrak{g}}$. It follows from Proposition 3.17 that $\mathring{\mathfrak{g}} = \mathfrak{g}(\mathring{A})$ is a simple finite-dimensional Lie algebra. The set $\mathring{\Delta} = \Delta \cap \mathring{\mathfrak{h}}^*$ is the root system of $\mathring{\mathfrak{g}}$; it is finite and consists of real roots (again by Proposition 3.17), the set $\mathring{\Delta}_+ = \mathring{\Delta} \cap \Delta_+$ being the set of positive roots. Denote by $\mathring{\Delta}_s$ and $\mathring{\Delta}_\ell$ the sets of short and long roots, respectively, in $\mathring{\Delta}$. Put $\mathring{Q} = \mathbf{Z}\mathring{\Delta}$. Let $\mathring{W}$ be the Weyl group of $\mathring{\Delta}$.

3.8.2 Relating $\mathfrak{g}$ and $\mathring{\mathfrak{g}}$

Recall the classification of imaginary and positive imaginary roots of $\mathfrak{g}$, following Theorem 3.20(b). We now characterize the set of real roots Δ^{re} and positive real roots Δ_+^{re} *in terms of* $\mathring{\Delta}$ *and* δ.

Proposition 3.24. *(a)* $\Delta^{\text{re}} = \left\{\alpha + n\delta : \alpha \in \mathring{\Delta}, n \in \mathbf{Z}\right\}$ *if* $r = 1$.

(b) $\Delta^{\text{re}} = \{\alpha + n\delta : \alpha \in \alpha\Delta_s, n \in \mathbf{Z}\} \cup \{\alpha + nr\delta : \alpha \in \alpha\Delta_\ell, n \in \mathbf{Z}\}$ *if* $r = 2$ *or* 4, *but* A *is not of type* $A^(2)_{2\ell}$.[4]

(c) If A *is of type* $A_{2\ell}^{(2)}$, *then*

$$\begin{aligned}\Delta^{\text{re}} = &\left\{\frac{1}{2}(\alpha + (2n-1)\delta : \alpha \in \mathring{\Delta}_\ell, n \in \mathbf{Z}\right\} \cup \\ &\left\{\alpha + n\delta : \alpha \in \mathring{\Delta}_s, n \in \mathbf{Z}\right\} \cup \\ &\left\{\alpha + 2n\delta : \alpha \in \mathring{\Delta}_\ell, n \in \mathbf{Z}\right\}.\end{aligned} \tag{3.42}$$

(d) $\Delta^{\text{re}} + r\delta = \Delta^{\text{re}}$.

(e) $\Delta_+^{\text{re}} = \{\alpha \in \Delta^{\text{re}}$ *with* $n > 0\} \cup \mathring{\Delta}_+$.

Remark. *There is an orthogonal decomposition of the coroot lattice as*

$$Q^\vee = \mathring{Q}^\vee \oplus \mathbf{Z}K, \tag{3.43}$$

and an isomorphism of lattices equipped with bilinear forms

$$Q^\vee(A) \simeq Q\left({}^tA\right). \tag{3.44}$$

Thusfar, it has been possible to describe the structure of the affine Kac-Moody algebra $\mathfrak{g}$ in terms of a finite-dimensional Kac-Moody algebra $\mathring{\mathfrak{g}}$. We extend this progress to a characterization of the Weyl group of the affine algebra $\mathfrak{g}$. Recall that W is generated by fundamental reflections $r_0, r_1, \ldots, r_\ell$ which act on $\mathfrak{h}^*$ by

$$r_i(\lambda) = \lambda - \left\langle \lambda, \alpha_i^\vee \right\rangle, \quad \lambda \in \mathfrak{h}^*. \tag{3.45}$$

As $\left\langle \delta, \alpha_i^\vee \right\rangle = 0$ for $i = 0, \ldots, \ell$, we have that

$$w(\delta) = \delta \forall w \in W. \tag{3.46}$$

Recall that the invariant standard form is also W-invariant.

Denote by $\mathring{W}$ the subgroup of W generated by $r_1, \ldots, r_\ell$. As $r_i(\Lambda_0) = \Lambda_0$ for $i = 1, \ldots, \ell$, $\mathring{W}$ acts trivially on $\mathbf{C}\Lambda_0 + \mathbf{C}\delta$; $\mathring{\mathfrak{h}}^*$ is also $\mathring{W}$-invariant. We conclude that $\mathring{W}$ acts faithfully on $\mathring{\mathfrak{h}}^*$, and we can identify $\mathring{W}$ with the Weyl group of the Lie algebra $\mathring{\mathfrak{g}}$, operating on $\mathring{\mathfrak{h}}^*$. By Proposition 3.14(e), $\left|\mathring{W}\right| < \infty$.

[4] c.f. the note attached to Proposition 3.23(d).

Recall that for a real root α, we have the reflection $r_\alpha \in W$ given by

$$r_\alpha(\lambda) = -\lambda - \left\langle \lambda, \alpha^\vee \right\rangle, \quad \lambda \in \mathfrak{h}^*. \tag{3.47}$$

Form the distinguished element

$$\theta := \delta - a_0\alpha_0 = \sum_{i=1}^{\ell} a_i\alpha_i \in \mathring{Q}, \tag{3.48}$$

where the a_i are as in Theorem 3.20(b). Given any $\alpha \in \mathring{\mathfrak{h}}^*$, we can form an operator on t_α on $\mathfrak{h}^*$ given by, for any $\lambda \in \mathfrak{h}^*$,

$$t_\alpha(\lambda) = \lambda + \langle \lambda, K\rangle\alpha - \left((\lambda, \alpha) + \frac{1}{2}|\alpha|^2\langle \alpha, K\rangle \right)\delta. \tag{3.49}$$

Now form the distinguished lattice $M \subset \mathring{\mathfrak{h}}^*_{\mathbf{R}}$. Let $\mathbf{Z}(\mathring{W} \cdot \theta^\vee)$ denote the lattice in $\mathring{\mathfrak{h}}_{\mathbf{R}}$ spanned over $\mathbf{Z}$ by the (finite) set $\mathring{W} \cdot \theta^\vee$, where $\theta^\vee$ denotes the θ element from the dual algebra $\mathfrak{g}\left({}^tA \right)$, and set $M = \nu(\mathbf{Z}(\mathring{W} \cdot \theta^\vee))$. M has the following properties.

$$\begin{aligned} &M = \overline{Q} = \mathring{Q} \text{ if } A \text{ is symmetric or } r > a_0; \\ &M = \nu(\overline{Q^\vee}) = \nu(\mathring{Q}^\vee) \text{ otherwise.} \end{aligned} \tag{3.50}$$

The lattice M, considered as an abelian group, acts faithfully on $\mathfrak{h}^*$ by (3.49). We call the corresponding subgroup of $GL(\mathfrak{h}^*)$ the *group of translations* and denote it by T. We may finally state the decomposition of the Weyl group W of $\mathfrak{g}$ in terms of the Weyl group $\mathring{W}$ of $\mathring{\mathfrak{g}}$.

Proposition 3.25. $W = \mathring{W} \ltimes T$.

Hence, we can obtain a rich description of the root and coroot lattices, the root system, and the Weyl group of an affine algebra $\mathfrak{g}$ in terms of the corresponding objects for the 'underlying' simple finite-dimensional Lie algebra $\mathring{\mathfrak{g}}$.

4 Highest-weight modules over Kac-Moody algebras

So far, we have emphasized the construction and structure of Kac-Moody algebras, and paid less attention to their representation theory. The representation theory of Kac-Moody algebras is a rich subject with many surprising connections to other areas of mathematics. In this section, we will introduce the notions of highest-weight modules over Kac-Moody algebras, and the category $\mathcal{O}$ and its Verma modules. We follow [Kac94, Ch. 9].

We start with an arbitrary complex $n \times n$ matrix A and consider the associated Lie algebra $\mathfrak{g}(A)$. We have the usual triangular decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{4.1}$$

and the corresponding decomposition of the universal enveloping algebra

$$\mathfrak{U}(\mathfrak{g}(A)) = \mathfrak{U}(\mathfrak{n}_-) \otimes \mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{n}_+). \tag{4.2}$$

Recall that a $\mathfrak{g}(A)$-module V is called $\mathfrak{h}$-diagonalizable if it admits a weight space decomposition $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ by weight spaces V_λ (c.f. §3.4).

Definition. *A nonzero vector from* V_λ *is called a* weight vector *of* weight λ.

Let $P(V) = \{\lambda \in \mathfrak{h}^* : V_\lambda \neq 0\}$ denote the set of weights of V. Finally, for $\lambda \in \mathfrak{h}^*$, set $D(\lambda) = \{\mu \in \mathfrak{h}^* : \mu \leqslant \lambda\}$.

Definition. *The category $\mathcal{O}$ is defined as follows. Objects in $\mathcal{O}$ are $\mathfrak{g}(A)$-modules V which are $\mathfrak{h}$-diagonalizable with finite-dimensional weight spaces and such that there exists a finite number of elements $\lambda_1, \ldots, \lambda_s \in \mathfrak{h}^*$ such that*

$$P(V) \subset \bigcup_{i=1}^{s} D(\lambda_i). \tag{4.3}$$

Morphisms in $\mathcal{O}$ are homomorphisms of $\mathfrak{g}(A)$-modules.

It follows from the following general fact that any submodule or quotient module of a module in $\mathcal{O}$ is in $\mathcal{O}$.

Proposition 4.1. *Let $\mathfrak{h}$ be a commutative Lie algebra, V a diagonalizable $\mathfrak{h}$-module, i.e.*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \text{ where } V_\lambda = \{v \in V : h(v) = \lambda(h)v \forall h \in \mathfrak{h}\}. \tag{4.4}$$

Then any submodule U of V is graded with respect to the gradation in (4.4).

It is clear that a sum or tensor product of a finite number of modules from $\mathcal{O}$ is again in $\mathcal{O}$. It is likewise the case that every module from $\mathcal{O}$ is restricted.

Definition. *A $\mathfrak{g}(A)$-module V is called a* highest-weight module *with* highest weight $\Lambda \in \mathfrak{h}^*$ *if there exists a nonzero vector $v_\Lambda \in V$ such that*

$$\mathfrak{n}_+(v_\Lambda) = 0, \quad h(v_\Lambda) = \Lambda(h)v_\Lambda \text{ for } h \in \mathfrak{h}, \quad \text{and } \mathfrak{U}(\mathfrak{g}(A))(v_\Lambda) = V. \tag{4.5}$$

The vector v_Λ is called a highest weight vector.

By the tensor product decomposition of the universal enveloping, we may restate the last condition as

$$\mathfrak{U}(\mathfrak{n}_-)(v_\Lambda) = V. \tag{4.6}$$

It follows that

$$V = \bigoplus_{\lambda \leqslant \Lambda} V_\lambda; \quad V_\Lambda = \mathbf{C}v_\Lambda; \quad \dim V_\lambda < \infty. \tag{4.7}$$

Therefore, a highest-weight module lies in $\mathcal{O}$, and every two highest-weight vectors are proportional.

Definition. *A $\mathfrak{g}(A)$-module $M(\Lambda)$ with highest weight Λ is called a* Verma module *if every $\mathfrak{g}(A)$-module with highest Λ is a quotient of $M(\Lambda)$.*

Proposition 4.2. *(a) For every $\Lambda \in \mathfrak{h}^*$, there exists a unique (up to isomorphism) Verma module $M(\Lambda)$.*

(b) Viewed as a $\mathfrak{U}(\mathfrak{n}_-)$-module, $M(\Lambda)$ is a free module of rank 1 generated by a highest-weight vector.

(c) $M(\Lambda)$ contains a unique proper maximal submodule $M'(\Lambda)$.

4.1 Inductive Verma construction

Recall the definition of an induced module. That is, let V be a left module over a Lie algebra $\mathfrak{a}$, and suppose there is a Lie algebra homomorphism $\mathfrak{a} \to \mathfrak{b}$. Then, by identifying $\mathfrak{a}$ with its image in $\mathfrak{b}$, we have that the induced $\mathfrak{b}$-module is given by

$$\mathfrak{U}(\mathfrak{b}) \otimes_{\mathfrak{U}(\mathfrak{a})} V, \tag{4.8}$$

where the homomorphism from $\mathfrak{a}$ to $\mathfrak{b}$ is used to translate between the left action on V and on $\mathfrak{U}(\mathfrak{b})$ in the bilinear tensor product. Here, the action of $\mathfrak{b}$ is induced by left multiplication in $\mathfrak{U}(\mathfrak{b})$. Then, we may construct a Verma module as an induced $(\mathfrak{n}_+\mathfrak{h})$-module as follows. Define the left $(\mathfrak{n}_+\mathfrak{h})$-module $\mathbf{C}_\lambda$ with underlying space $\mathbf{C}$ by $\mathfrak{n}_+(1) = 0$, $h(1) = \langle \lambda, h \rangle 1$ for $h \in \mathfrak{h}$. Then

$$M(\lambda) = \mathfrak{U}(\mathfrak{g}(A)) \otimes_{\mathfrak{U}(\mathfrak{n}_+ + \mathfrak{h})} \mathbf{C}_\lambda. \tag{4.9}$$

4.2 Primitive vectors and weights

It follows from Proposition 4.2(c) that among the modules with highest weight Λ there is a unique irreducible one, namely

$$L(\Lambda) := M(\Lambda)/M'(\Lambda). \tag{4.10}$$

It is evident that $L(\Lambda)$ is a quotient of any module with highest weight Λ. To show that the modules $L(\Lambda)$ exhaust all irreducible modules from the category $\mathcal{O}$, as well as for other purposes, we make the following definitions. Let V be a $\mathfrak{g}(A)$-module.

Definition. *A vector $v \in V_\lambda$ is called* primitive *if there exists a submodule $U \subset V$ such that*

$$v \notin U; \quad \mathfrak{n}_+(v) \subset U. \tag{4.11}$$

Then λ is called a primitive weight.

One defines primitive vectors and weights for a $\mathfrak{g}'(A)$-module similarly. A weight vector v such that $\mathfrak{n}_+(v) = 0$ is obviously primitive.

Proposition 4.3. *Let V be a nonzero module from the category $\mathcal{O}$. Then*

- *(a) V contains a nonzero weight vector v such that $\mathfrak{n}_+(v) = 0$.*
- *(b) The following conditions are equivalent.*
  - *(i) V is irreducible;*
  - *(ii) V is a highest weight module and any primitive vector of V is a highest-weight vector;*
  - *(iii) $V \simeq L(\Lambda)$ for some $\Lambda \in \mathfrak{h}^*$.*
- *(c) V is generated by its primitive vectors as a $\mathfrak{g}(A)$-module.*

Thus, we have a bijection between $\mathfrak{h}^*$ and irreducible modules from the category $\mathcal{O}$ given by $\Lambda \mapsto L(\Lambda)$. Note that $L(\Lambda)$ can also be defined as an irreducible $\mathfrak{g}(A)$-module which admits a nonzero vector v_Λ such that

$$\mathfrak{n}_+(v_\Lambda) = 0 \text{ and } h(v_\Lambda) = \Lambda(h)v_\Lambda \text{ for } h \in \mathfrak{h}. \tag{4.12}$$

Henceforth, let $\mathfrak{U}_0(\mathfrak{g})$ denote the augmentation ideal $\mathfrak{g}\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{U}(\mathfrak{g})$. We have the following 'Schur-style' lemma.

Lemma 4.4. $\mathrm{End}_{\mathfrak{g}(A)} L(\Lambda) = \mathbf{C} I_{L(\Lambda)}$.

Let $L(\Lambda)^*$ now e the $\mathfrak{g}(A)$-module dual to $L(\Lambda)$. Then $L(\Lambda)^* = \prod_\lambda (L(\Lambda)_\lambda)^*$. The subspace

$$L^*(\Lambda) := \bigoplus (L(\Lambda)_\lambda)^* \tag{4.13}$$

is a submodule of the $\mathfrak{g}(A)$-module $L(\Lambda)^*$. It is clear that the module $L^*(\Lambda)$ is irreducible, and that for $\nu \in (L(\Lambda)_\Lambda)^*$ one has

$$\mathfrak{n}_-(\nu) = 0; \quad h(\nu) = -\langle \Lambda, h\rangle \nu \text{ for } h \in \mathfrak{h}. \tag{4.14}$$

Such a module is called an *irreducible module with lowest weight* $-\Lambda$. As before, we have a bijection between $\mathfrak{h}^*$ and irreducible lowest-weight modules given by $\Lambda \mapsto L^*(-\Lambda)$.

Denote by π_Λ the action of $\mathfrak{g}(A)$ on $L(\Lambda)$, and introduce the new action π^*_Λ on the space $L(\Lambda)$ by

$$\pi^*_\Lambda(g)\nu = \pi_\Lambda(\omega(g))\nu, \tag{4.15}$$

where ω is the Chevalley involution of $\mathfrak{g}(A)$. It is clear that $(L(\Lambda), \pi^*_\Lambda)$ is an irreducible $\mathfrak{g}(A)$-module with lowest weight $-\Lambda$. By the uniqueness theorem, this module can be identified with $L^*(\Lambda)$, and the pairing between $L(\Lambda)$ and $L^*(\Lambda)$ gives us a nondegenerate bilinear form B on $L(\Lambda)$ such that

$$B(g(x), y) = -B(x, \omega(g)(y)) \forall g \in \mathfrak{g}(A) \text{ and } x, y \in L(\Lambda). \tag{4.16}$$

A bilinear form on $L(\Lambda)$ which satisfies (4.16) is called a *contravariant* bilinear form.

Proposition 4.5. *Every $\mathfrak{g}(A)$-module $L(\Lambda)$ carries a unique-up-to-constant-factor nondegenerate contravariant bilinear form B. The form is symmetric and $L(\Lambda)$ decomposes into an orthogonal direct sum of weight spaces with respect to this form.*

4.3 Complete reducibility

We now may attack the question of reducibility.

Lemma 4.6. *Let V be a $\mathfrak{g}(A)$-module from the category $\mathcal{O}$. If for any two primitive weights λ and μ of V the inequality $\lambda \geqslant \mu$, then the module V is completely reducible.*

It is not the case that each module $V \in \mathcal{O}$ admits a composition series (a sequence of submodules $V \supset V_1 \supset V_2 \supset \cdots$ such that each V_i/V_{i+1} is irreducible. However, there is a kind of substitute for this property that does hold.

Lemma 4.7. *Let $V \in \mathcal{O}$ and $\lambda \in \mathfrak{h}^*$. Then there exists a filtration by a sequence of submodules $V = V_t \supset V_{t-1} \supset \cdots \supset V_1 \supset V_0 = 0$ and a subset $J \subset \{1, \ldots, t\}$ such that*

(i) if $j \in J$, then $V_j/V_{j-1} \simeq L(\lambda_j)$ for some $\lambda_j \geqslant \lambda$.

(ii) if $j \notin J$, then $(V_j/V_{j-1})_\mu = 0$ for every $\mu \geqslant \lambda$.

Let $V \in \mathcal{O}$ and $\mu \in \mathfrak{h}^*$. Fix a $\lambda \in \mathfrak{h}^*$ such that $\mu \geqslant \lambda$, and construct a filtration as per Lemma 4.7.

Definition. *Let the* multiplicity *of $L(\mu)$ in V, denoted by $[V : L(\mu)]$, be the number of times μ appears among $\{\lambda_j : j \in J\}$, which is independent of the filtration and the choice of λ.*

Remark. *$L(\mu)$ has a nonzero multiplicity in V if and only if μ is a primitive weight of V.*

4.4 Formal characters

To study the formal characters of modules from $\mathcal{O}$, we define an algebra $\mathcal{E}$ over $\mathbf{C}$ in the following way. Elements of $\mathcal{E}$ are sums

$$\sum_{\lambda \in \mathfrak{h}^*} c_\lambda e(\lambda), \tag{4.17}$$

where $c_\lambda \in \mathbf{C}$ and $c_\lambda = 0$ for λ outside the union of a finite number of sets of the form $D(\lambda)$. $\mathcal{E}$ becomes a commutative associative algebra if we set $e(\lambda)e(\mu) = e(\lambda + \mu)$ and extend by linearity in the usual way. $e(0)$ is our identity.

Definition. *The elements* $e(\lambda)$ *are called* formal exponents.

Formal exponents are linearly independent and in one-to-one correspondence with the elements $\lambda \in \mathfrak{h}^*$.

Now fix V a module from the category $\mathcal{O}$ and let $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ be its weight space decomposition.

Definition. *Define the* formal character *of* V *by*

$$\operatorname{ch} V = \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda)\, e(\lambda) \in \mathcal{E}. \tag{4.18}$$

Proposition 4.8. *Let* V *be a* $\mathfrak{g}(A)$*-module from* $\mathcal{O}$. *Then*

$$\operatorname{ch} V = \sum_{\lambda \in \mathfrak{h}^*} [V : L(\lambda)] \operatorname{ch} L(\lambda). \tag{4.19}$$

We record now a formula for the formal character of a Verma module $M(\Lambda)$.

Lemma 4.9.

$$\operatorname{ch} M(\Lambda) = e(\Lambda) \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{-\operatorname{mult} \alpha}. \tag{4.20}$$

Suppose now that A is a symmetrizable matrix and let $(\ ,\ )$ be a bilinear form on $\mathfrak{g}(A)$. Then the generalized Casimir operator Ω acts on each module from $\mathcal{O}$.

Lemma 4.10. *(a) If* V *is a* $\mathfrak{g}(A)$*-module with highest weight* Λ, *then*

$$\Omega = \left(|\Lambda + \rho|^2 - |\rho|^2\right) I_V. \tag{4.21}$$

(b) If V *is a module from the category* $\mathcal{O}$ *and* v *is a primitive vector with weight* λ, *then there exists a submodule* $U \subset V$ *such that* $v \notin U$ *and*

$$\Omega(v) = \left(|\lambda + \rho|^2 - |\rho|^2\right) v \mod U. \tag{4.22}$$

Proposition 4.11. *Let* V *be a* $\mathfrak{g}(A)$*-module with highest weight* Λ. *Then*

$$\operatorname{ch} V = \sum_{\substack{\lambda \leqslant \Lambda \\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} c_\lambda \operatorname{ch} M(\lambda), \text{ where } c_\lambda \in \mathbf{Z}, c_\lambda = 1. \tag{4.23}$$

4.5 Reducibility

Staying in the symmetrizable case, we now use the Casimir operator to study questions of irreducibility and complete reducibility in $\mathcal{O}$.

Proposition 4.12. *Let A be a symmetrizable matrix.*

(a) If $2(\Lambda+\rho,\beta) \neq (\beta,\beta) \forall 0 \neq \beta \in Q_+$, then the $\mathfrak{g}(A)$-module $M(\Lambda)$ is irreducible.

(b) If V is a $\mathfrak{g}(A)$-module from the category $\mathcal{O}$ such that for any two primitive weights λ and μ of V, such that $\lambda-\mu=\beta>0$, one has $2(\lambda+\rho,\beta) \neq (\beta,\beta)$, then V is completely reducible.

Now we consider the derived subalgebra $\mathfrak{g}'(A) = [\mathfrak{g}(A),\mathfrak{g}(A)]$ of $\mathfrak{g}(A)$ instead of $\mathfrak{g}(A)$, for a general A (not necessarily symmetrizable). Recall that $\mathfrak{g}(A) = \mathfrak{g}'(A)+\mathfrak{h}$, and that $\mathfrak{h}' = \sum_i \mathbf{C}\alpha_i^\vee = \mathfrak{g}'(A)\cap\mathfrak{h}$. Recall the free abelian group Q. We have the following decomposition of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}(A))$ with respect to $\mathfrak{h}$:

$$\mathfrak{U}(\mathfrak{g}(A)) = \bigoplus_{\beta\in Q} \mathfrak{U}_\beta, \text{ where} \tag{4.24}$$

$$U_\beta := \{x \in \mathfrak{U}(\mathfrak{g}(A)) : [h,x] = \langle\beta,h\rangle x \forall h \in \mathfrak{h}\}. \tag{4.25}$$

Put $U'_\beta = \mathfrak{U}(\mathfrak{g}'(A)) \cap U_\beta$, so that $\mathfrak{U}(\mathfrak{g}'(A)) = \bigoplus_\beta U'_\beta$.

Definition. *A $\mathfrak{g}'(A)$-module V is called a* highest-weight module *with* highest weight $\Lambda \in (\mathfrak{h}')^*$ *if V admits a Q_+-gradation $V = \bigoplus_{\beta\in Q_+} V_{\Lambda-\beta}$ such that $U'_\beta(V_{\Lambda-\alpha}) \subset V_{\Lambda-\alpha+\beta}$*, $\dim V_\Lambda = 1$, $h(v) = \Lambda(h)v$ *for* $h \in (\mathfrak{h}')^*$, $v \in V_\Lambda$, *and* $V = \mathfrak{U}(\mathfrak{g}'(A))(V_\Lambda)$.

In other words, this is a restriction of a highest-weight module over $\mathfrak{g}(A)$ to $\mathfrak{g}'(A)$. We define the Verma module $M(\Lambda)$ over $\mathfrak{g}'(A)$ in the same way and show that it contains a unique proper maximal garded submodule $M'(\Lambda)$. We put $L(\Lambda) = M(\Lambda)/M'(\Lambda)$, which is a restriction of an irreducible $\mathfrak{g}(A)$-module to $\mathfrak{g}'(A)$.

Lemma 4.13. *The $\mathfrak{g}'(A)$-module $L(\Lambda)$ is irreducible.*

Definition. *The* labels *of $\Lambda \in \mathfrak{h}^*$ are the values $\langle\Lambda,\alpha_i^\vee\rangle$ ($i=1,\ldots,n$).*

If $\Lambda, M \in \mathfrak{h}^*$ have the same labels, they may differ only off $\mathfrak{h}'$. Then Lemma 4.13 implies that the modules $L(\Lambda)$ and $L(M)$, when restricted to $\mathfrak{g}'(A)$, are irreducible and isomorphic as $\mathfrak{g}'(A)$-modules, and that the actions of elements of $\mathfrak{g}(A)$ on them differ only by scalar operations. Finally, note that

$$\dim L(\Lambda) = 1 \Leftrightarrow \Lambda|_{\mathfrak{h}'} = 0. \tag{4.26}$$

Now we return to the case where A is symmetrizable. We may fix a symmetric invariant bilinear form on $\mathfrak{g}'(A)$ which is defined also on $\mathfrak{h}'$, and whose kernel is $\mathfrak{c}$. Put $(\beta,\gamma) = (\nu^{-1}(\beta),\nu^{-1}(\gamma))$ for $\beta,\gamma \in Q$, and define $\rho \in (\mathfrak{h}')^*$ by $\langle\rho,\lambda_i^\vee\rangle = \frac{1}{2}a_{ii}$ for $i=1,\ldots,n$.

Proposition 4.14. *Let A be a (possibly infinite) symmetrizable matrix.*

(a) If $2\langle\Lambda+\rho,\nu^{-1}(\beta)\rangle \neq (\beta,\beta) \forall \beta \in \mathbf{Q}_+ - \{0\}$, then the $\mathfrak{g}'(A)$-module $M(\Lambda)$ is irreducible.

(b) Let V be a $\mathfrak{g}'(A)$-module such that the following conditions are satisfied:

(i) $e_i(v) = 0 \forall v \in V$ and all but a finite number of the e_i;

(ii) $\forall v \in V \exists k > 0 : e_{i_1}\cdots e_{i_s}(v) = 0$ whenever $s > k$;

(iii) $V = \bigoplus_{\lambda\in(\mathfrak{h}')^*} V_\lambda$, where $V_\lambda = \{v \in V : h(v) = \langle\lambda,h\rangle v \forall h \in \mathfrak{h}'\}$;

(iv) if λ and $\mu \in (\mathfrak{h}')^*$ are primitive weights such that $\lambda-\mu = \beta|_{\mathfrak{h}'}$ for some $\beta \in Q_+ - \{0\}$, then $2\langle\lambda+\rho,\nu^{-1}(\beta)\rangle \neq (\beta,\beta)$.

Then V is completely reducible; i.e. is isomorphic to a direct sum of $\mathfrak{g}'(A)$-modules of the form $L(\Lambda)$, $\Lambda \in (\mathfrak{h}')^*$.

Corollary 4.15. *Let A be a symmetrizable matrix with nonpositive real entries and let V be a $\mathfrak{g}'(A)$-module satisfying conditions (i)–(iii) of Proposition 4.14(b). Suppose that for every weight λ of V one has $\langle \lambda, \alpha_i^\vee \rangle > 0 \forall i$. Then V is a direct sum of irreducible $\mathfrak{g}'(A)$-modules, which are free of rank 1 when viewed of $\mathfrak{U}(\mathfrak{n}_-)$-modules.*

We end our discussion of highest-weight modules over Kac-Moody algebras to mention that the theory of these modules has ready application to finding the defining generators and relations of symmetrizable Kac-Moody Lie algebras (c.f. [Kac94, §9.11–12]).

5 Physical applications of Kac-Moody algebras

We will conclude our exposition by discussing some applications to physics, namely the Sugawara construction, the Virasoro algebra, and the coset construction, which are fundamental to string theory and the basic constructions of conformal field theory. In this section, we follow [Kac94, Ch. 12].

5.1 Sugawara construction

The origin of the Sugawara construction in physics is Hirotaka Sugarawara's 1968 paper [Sug68]. At this time, quantum chromodynamics (QCD) was not yet developed, and Sugawara was working to find a theory of strong interactions that quantized the interaction in terms of 'currents.' His theory was superseded by QCD, but as it was one of the first applications of Kac-Moody algebras, we describe it here.

As part of the classification of affine Kac-Moody algebras, there is a split of *twisted* and *non-twisted* affine algebras. These two subcollections are classified by their Dynkin diagrams, and are treated in [Kac94, Ch. 8]. We suppose our affine algebra $\mathfrak{g}'$ is *untwisted* in what follows. We then have an isomorphism

$$\mathfrak{g}' \simeq \mathbf{C}[t, t^{-1}] \otimes \mathring{\mathfrak{g}} + \mathbf{C}K \tag{5.1}$$

with commutation relations

$$\left[x^{(m)}, y^{(n)}\right] = [x, y]^{(m+n)} + m\delta_{m,-n}(x, y)K. \tag{5.2}$$

Here $\mathring{\mathfrak{g}}$ is a simple finite-dimensional Lie algebra, (x, y) is the normalized invariant bilinear form on $\mathring{\mathfrak{g}}$, and $x^{(n)}$ stands for $t^n \otimes x$ ($n \in \mathbf{Z}, x \in \mathring{\mathfrak{g}}$).

Let $\{u_i\}$ and $\{u^i\}$ be dual bases of $\mathring{\mathfrak{g}}$; i.e. $(u_i, u^j) = \delta_{ij}$. Then the Casimir operator of $\mathring{\mathfrak{g}}$,

$$\mathring{\Omega} = \sum_i u_i u^i, \tag{5.3}$$

is independent of the choice of dual bases. Additionally, the Casimir operator

$$\Omega = 2(K + h^\vee)d + \mathring{\Omega} + 2\sum_{n=1}^{\infty}\sum_i u_i^{(-n)} u^{i(n)} \tag{5.4}$$

is the Casimir operator of the affine algebra

$$\mathfrak{g} = \mathfrak{g}' + \mathbf{C}d, \quad \text{where } \left[d, x^{(n)}\right] = nx^{(n)}. \tag{5.5}$$

Recall the definition of a restricted $\mathfrak{g}'$-module in section 3.3; i.e. any module V such that $\forall v \in V$, $x^{(j)}(v) = 0 \forall x \in \mathring{\mathfrak{g}}, j \gg 0$. Consider all series $\sum_{j=1}^{\infty} u_j$ with $u_j \in \mathfrak{U}(\mathfrak{g}')$, the universal enveloping algebra of $\mathfrak{g}'$, such that for any restricted $\mathfrak{g}'$-module V and any $v \in V$, $u_j(v) = 0$ for all but finitely mnay u_j. We identify two such series if they represent the same operator in every restricted $\mathfrak{g}'$-module, thereby obtaining an algebra $\mathfrak{U}_c(\mathfrak{g}')$ which contains $\mathfrak{U}(\mathfrak{g}')$ and acts on every restricted $\mathfrak{g}'$-module.

Definition. *Define the* restricted completion of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}')$ *to be this algebra* $\mathfrak{U}_c(\mathfrak{g}')$.

We are now in a position to introduce our main object of study. Let T_n ($n \in \mathbf{Z}$) be an operator given by

$$\begin{aligned} T_0 &= \sum_i u_i u^i + 2\sum_{n=1}^{\infty}\sum_i u_i^{(-n)}u^{i(n)}, \\ T_n &= \sum_{m\in\mathbf{Z}}\sum_i u_i^{(-m)}u^{i(m+n)} \quad \text{if } n \neq 0. \end{aligned} \tag{5.6}$$

Definition. *The operators* T_n *are the* Sugawara operators.

In fact, these operators are contained in $\mathfrak{U}_c(\mathfrak{g}')$, and are independent of the choice of dual bases of $\mathring{\mathfrak{g}}$. We have the following basic information about how the Sugawara operators act.

Lemma 5.1. *(a) For* $x \in \mathring{\mathfrak{g}}$ *and* $n, m \in \mathbf{Z}$*, one has*

$$\left[x^{(m)}, T_n\right] = 2(K + h^\vee)mx^{(m+n)}. \tag{5.7}$$

(b) Let V *be a restricted* $\mathfrak{g}$*-module and let* $v \in V$ *be such that* $\mathfrak{n}_+(v) = 0$ *and* $h(v) = \langle\lambda, h\rangle v$ *for some* $\lambda \in \mathfrak{h}^*$*. Then*

$$T_0(v) = (\overline{\lambda}, \overline{\lambda} + 2\overline{\rho})v. \tag{5.8}$$

We now write down the commutator $[T_m, T_n]$ of two Sugawara operators.

Proposition 5.2.

$$[T_m, T_n] = 2(K + h^\vee)\left((m-n)T_{m+n} + \delta_{m,-n}\frac{m^3 - m}{6}(\dim\mathring{\mathfrak{g}})K\right). \tag{5.9}$$

Lemma 5.1 and Proposition 5.2 immediately yield the following

Corollary 5.3. *Let* V *be a restricted* $\mathfrak{g}'$*-module such that* K *is a scalar operator* kI, $k \neq -h^\vee$*. Let*

$$L_n = \frac{1}{2(k+h^\vee)}T_n, \quad n \in \mathbf{Z} \tag{5.10}$$

$$c(k) = \frac{k(\dim\mathring{\mathfrak{g}})}{k + h^\vee}, \tag{5.11}$$

$$h_\Lambda = \frac{(\Lambda + 2\rho, \Lambda)}{2(k+h^\vee)} \quad \textit{if } V = L(\lambda), \tag{5.12}$$

where $\Lambda \in \mathfrak{h}^*$ *is a highest weight, and* $L(\Lambda)$ *is the unique irreducible module with highest weight* Λ *(c.f. [Kac94, §9.2]).*

(a) Letting

$$d_n \mapsto L_n, c \mapsto c(k) \tag{5.13}$$

extends V *to a module over* $\mathfrak{g}' + \text{Vir}$*, in particular,* V *extends to a module over* $\mathfrak{g}$ $(= \mathfrak{g}' + \mathbf{C}d)$.

(b) If V *is the* $\mathfrak{g}$*-module* $L(\Lambda)$*, then* $L_0 = h_\lambda I - d$.

Definition. *The number* $c(k)$ *is called the* conformal anomoly *of the* $\mathfrak{g}'$*-module* V*, and the number is called the* vacuum anomoly *of* $L(\Lambda)$.

The appearance of Vir in the above corollary requires some explanation.

5.2 Virasoro algebra

Let $\mathring{\mathfrak{g}}$ be a simple finite-dimensional Lie algebra, put $\mathcal{L} = \mathbf{C}[t, t^{-1}]$, and consider the *loop algebra*

$$\mathcal{L}(\mathring{\mathfrak{g}}) := \mathcal{L} \otimes_{\mathbf{C}} \mathring{\mathfrak{g}}. \tag{5.14}$$

This is an infinite-dimensional Lie algebra with bracket $[\ ,\ ]_0$ given by

$$[P \otimes x, Q \otimes y]_0 = PQ \otimes [x, y] \quad (P, Q \in \mathcal{L}; x, y \in \mathring{\mathfrak{g}}). \tag{5.15}$$

Denote by $\tilde{\mathcal{L}}(\mathring{\mathfrak{g}})$ the extension of the Lie algebra $\mathcal{L}(\mathring{\mathfrak{g}})$ by a 1-dimensional center; explicitly $\tilde{\mathcal{L}}(\mathring{\mathfrak{g}}) := \mathcal{L}(\mathring{\mathfrak{g}}) \oplus \mathbf{C}K$, with bracket

$$[a + \lambda K, b + \mu K] = [a, b]_0 + \psi(a, b)K \quad (a, b \in \mathcal{L}(\mathring{\mathfrak{g}}); \lambda, \mu \in \mathbf{C}), \tag{5.16}$$

where ψ is the $\mathbf{C}$-valued 2-cocycle on the Lie algebra $\mathcal{L}(\mathring{\mathfrak{g}})$

$$\psi(a, b) := \operatorname{res}\left(\frac{da}{dt}, b\right)_t, \tag{5.17}$$

where res is the usual residue, a linear functional on $\mathcal{L}$, and $(\ ,\ )_t$ is the extension of a nondegenerate invariant symmetric bilinear complex-valued form $(\ ,\ )$ on $\mathring{\mathfrak{g}}$ to an $\mathcal{L}$-valued bilinear form on $\mathcal{L}(\mathring{\mathfrak{g}})$ by

$$(P \otimes x, Q \otimes y)_t = PQ(x, y). \tag{5.18}$$

Also, we extend every derivation D of the algebra $\mathcal{L}$ to a derivation of the Lie algebra $\mathcal{L}(\mathring{\mathfrak{g}})$ by

$$D(P \otimes x) = D(P) \otimes x. \tag{5.19}$$

Then, let d_s denote the endomorphism of the space $\tilde{\mathcal{L}}(\mathring{\mathfrak{g}})$ defined by

$$d_s|_{\mathcal{L}(\mathring{\mathfrak{g}})} = -t^{s+1}\frac{d}{dt}, \quad d_s(K) = 0. \tag{5.20}$$

Proposition 5.4. *d_s is a derivation of $\tilde{\mathcal{L}}(\mathring{\mathfrak{g}})$.*

Note that

$$\mathfrak{d} := \bigoplus_{j \in \mathbf{Z}} \mathbf{C} d_j \tag{5.21}$$

is a $\mathbf{Z}$-graded subalgebra in der $\tilde{\mathcal{L}}(\mathring{\mathfrak{g}})$ with the commutation relations

$$[d_i, d_j] = (i - j)d_{i+j}. \tag{5.22}$$

This is the Lie algebra of regular vector fields on $\mathbf{C}^\times$ (i.e. derivations of $\mathcal{L}$). The Lie algebra $\mathfrak{d}$ has a nontrivial central extension by a 1-dimensional center, say $\mathbf{C}c$, that is unique up to isomorphism.

Definition. *The unique-up-to-isomorphism nontrivial central extension $\mathfrak{d} \oplus \mathbf{C}c$ of $\mathfrak{d}$ is called the* Virasoro algebra Vir, *and is defined by the commutation relations*

$$[d_i, d_j] = (i - j)d_{i+j} + \frac{1}{12}\left(i^3 - i\right) d_{i,-j} c, \quad i, j \in \mathbf{Z}. \tag{5.23}$$

5.3 Virasoro operators

We now may extend the Sugawara construction to the case of a reductive finite-dimensional Lie algebra $\mathfrak{g}$ (we use this notation in place of $\mathring{\mathfrak{g}}$ for this section). We have the decomposition of $\mathfrak{g}$ into a direct sum of ideals as

$$\mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)} \oplus \mathfrak{g}_{(2)} \oplus \cdots , \tag{5.24}$$

where $\mathfrak{g}_{(0)}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}_{(i)}$ with $i \geqslant 1$ are simple. We fix on $\mathfrak{g}$ a non-degenerate invariant bilinear form $(\ ,\ )$ so that (5.24) is an orthogonal decomposition, and we assume that the restriction of $(\ ,\ )$ to each $\mathfrak{g}_{(i)}$ for $i \geqslant 1$ is the normalized invariant form. We shall call such a form a normalized invariant form on $\mathfrak{g}$. Put

$$\tilde{\mathcal{L}}(\mathfrak{g}) := \bigoplus_{i\geqslant 0} \tilde{\mathcal{L}}\left(\mathfrak{g}_{(i)}\right), \quad \text{where } \tilde{\mathcal{L}}\left(\mathfrak{g}_{(i)}\right) = \tilde{L}\left(\mathfrak{g}_{(i)}\right) + \mathbf{C}K_i \tag{5.25}$$

and

$$\hat{\mathcal{L}}(\mathfrak{g}) := \tilde{\mathcal{L}}(\mathfrak{g}) + \mathbf{C}d, \quad \text{where } d|_{\tilde{\mathcal{L}}(\mathfrak{g}_{(i)})} = -t\frac{d}{dt}, d\left(K_i\right) = 0. \tag{5.26}$$

Definition. *The Lie algebras $\tilde{\mathcal{L}}(\mathfrak{g})$ and $\hat{\mathcal{L}}(\mathfrak{g})$ are called affine algebras associated to the reductive Lie algebra $\mathfrak{g}$. The subalgebras $\tilde{\mathcal{L}}\left(\mathfrak{g}_{(i)}\right)$ (resp. $\tilde{\mathcal{L}}\left(\mathfrak{g}_{(i)}\right) + \mathbf{C}d$) are called* components *of $\tilde{\mathcal{L}}(\mathfrak{g})$ (resp. $\hat{\mathcal{L}}(\mathfrak{g})$).*

Remark. $\mathfrak{c} := \mathfrak{g}_{(0)} + \sum_{i\geqslant 1} \mathbf{C}K_i$ *is the center of $\tilde{\mathcal{L}}(\mathfrak{g})$ and $\hat{\mathcal{L}}(\mathfrak{g})$.*

We identify $\mathfrak{g}$ with the subalgebra $1 \otimes \mathfrak{g}$, let $\overline{\mathfrak{h}}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\mathfrak{g} = \overline{\mathfrak{n}}_- \oplus \overline{\mathfrak{h}} \oplus \overline{\mathfrak{n}}_+$ be a triangular decomposition of $\mathfrak{g}$. The subalgebra $\mathfrak{h} = \overline{\mathfrak{h}} + \mathfrak{c} + \mathbf{C}d$ is called the Cartan subalgebra of $\hat{\mathcal{L}}(\mathfrak{g})$. We also have a triangular decomposition

$$\hat{\mathcal{L}}(\mathfrak{g}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \text{ where} \tag{5.27}$$

$$\mathfrak{n}_- = \left(t^{-1}\mathbf{C}\left[t^{-1}\right] \otimes \left(\overline{\mathfrak{n}}_+ + \overline{\mathfrak{h}}\right)\right) + \mathbf{C}\left[t^{-1}\right] \otimes \overline{\mathfrak{n}}_-, \text{ and} \tag{5.28}$$

$$\mathfrak{n}_+ = \left(t\mathbf{C}[t] \otimes \left(\overline{\mathfrak{n}}_- + \overline{\mathfrak{h}}\right)\right) + \mathbf{C}[t] \otimes \overline{\mathfrak{n}}_+. \tag{5.29}$$

For $\lambda \in \mathfrak{h}^*$, we denote its restriction to $\overline{\mathfrak{h}}$ by $\overline{\lambda}$. We define $\delta \in \mathfrak{h}^*$ by

$$\delta|_{\overline{\mathfrak{h}}+\mathfrak{c}} = 0, \quad \langle \delta, d\rangle = 1. \tag{5.30}$$

Given $\Lambda \in \mathfrak{h}^*$, there is a unique irreducible $\hat{\mathcal{L}}(\mathfrak{g})$-module that admits a non-zero vector v_Λ such that $\mathfrak{n}_+\left(v_\lambda\right) = 0$ and $h\left(v_\Lambda\right) = \langle \Lambda, h\rangle v_\Lambda$ for $h \in \mathfrak{h}$. By the uniqueness of $L(\Lambda)$, we have

$$L(\Lambda) = \bigotimes_{i\geqslant 0} L\left(\Lambda_{(i)}\right), \tag{5.31}$$

where $\Lambda_{(i)}$ denotes the restriction of Λ to $\mathfrak{h}_{(i)} := \mathfrak{h} \cap \hat{\mathcal{L}}\left(\mathfrak{g}_{(i)}\right)$ and $\mathcal{L}\left(\Lambda_{(i)}\right)$ is the $\hat{\mathcal{L}}\left(\mathfrak{g}_{(i)}\right)$-module with highest weight $\Lambda_{(i)}$.

We let k_i, the eigenvalue of K_i on $L(\Lambda)$, be the i*th level* of Λ, and let $k = (k_0, k_1, \ldots)$. Define

$$c(k) := \sum_i c\left(k_i\right), \tag{5.32}$$

$$h_\Lambda := \sum_i h_{\Lambda_{(i)}} \tag{5.33}$$

$$m_\Lambda = \sum_i m_{\Lambda_{(i)}}. \tag{5.34}$$

Let V be a restricted $\tilde{\mathcal{L}}(\mathfrak{g})$-module such that K_i acts as $k_i I$ and k_i is distinct from the dual Coxeter number of $\tilde{\mathcal{L}}\left(\mathfrak{g}_{(i)}\right)$, which we will denote by $h_i^\vee$, and which is generically defined as

$$h := \sum_{i=0}^{\ell} a_i, \tag{5.35}$$

where ℓ is the rank of the generic affine algebra and a_i are the labels of the Dynkin diagram as before, and $a_i^\vee$ those of the diagram associated to the dual algebra or transpose matrix. Let $T_n^{(i)}$ be the Sugawara operators for $\tilde{\mathcal{L}}\left(\mathfrak{g}_{(i)}\right)$, and let, $\forall n \in \mathbf{Z}$,

$$L_n^{(i)} := \frac{1}{2\left(k_i + h_i^\vee\right)} T_n^{(i)}, \quad L_n^{\mathfrak{g}} := \sum_i L_n^{(i)}. \tag{5.36}$$

Definition. *The operators* $L_n^{\mathfrak{g}}$ *are called the* Virasoro operators *for the* $\mathfrak{g}$*-module* V.

Letting $d_n \mapsto L_n^{\mathfrak{g}}$, $c \mapsto c(k)$ extends V to a module over $\tilde{\mathcal{L}}(\mathfrak{g}) + \text{Vir}$. We also have the formula

$$L_0^{\mathfrak{g}} = \sum_i \frac{\Omega_i}{2\left(k_i + h_i^\vee\right)} - d, \tag{5.37}$$

where Ω_i is the Casimir operator for $\hat{\mathcal{L}}\left(\mathfrak{g}_{(i)}\right)$.

5.4 Coset Virasoro construction

Let $\mathfrak{g}$ be a reductive finite-dimensional Lie algebra with a normalized invariant form $(\ ,\ )$, and let $\dot{\mathfrak{g}}$ be a reductive subalgebra of $\mathfrak{g}$ such that $(\ ,\ )|_{\dot{\mathfrak{g}}}$ is non-degenerate. Let

$$\mathfrak{g} = \bigoplus_{i \geqslant 0} \mathfrak{g}_{(i)} \quad \text{and} \quad \dot{\mathfrak{g}} = \bigoplus_{i \geqslant 0} \dot{\mathfrak{g}}_{(i)} \tag{5.38}$$

be the decompositions of $\mathfrak{g}$ and $\dot{\mathfrak{g}}$. Let $(\ ,\ )$ be a normalized invariant form on $\dot{\mathfrak{g}}$ that coincides with $(\ ,\ )$ on $\dot{\mathfrak{g}}_{(0)}$. Due to the uniqueness of the invariant bilinear form on a simple Lie algebra we have for $x, y \in \dot{\mathfrak{g}}_{(s)}$, $s \geqslant 1$ that

$$\left(x_{(r)}, y_{(r)}\right) = j_{sr}(x, y), \tag{5.39}$$

where $x_{(r)}$ denotes the projection of x on $\mathfrak{g}_{(r)}$ and j_{sr} is a (positive) number independent of x and y; we let $j_{0r} = 1$.

Definition. *The numbers* j_{sr} *($s, r \geqslant 0$) are called* Dynkin indices.

Let V be a restricted $\tilde{\mathcal{L}}(\mathfrak{g})$-module such that K_i acts as $k_i I$, $k_i \neq -h_i^\vee$. This is a $\tilde{\mathcal{L}}(\dot{\mathfrak{g}})$ module with $\dot{K}_i$ acting as $\dot{k}_i I$, where

$$\dot{k}_s = \sum_i j_{si} k_j. \tag{5.40}$$

Assume that $\dot{k}_i \neq -\dot{h}_i^\vee$. Put

$$L_n^{\mathfrak{g},\dot{\mathfrak{g}}} := L_n^{\mathfrak{g}} - L_n^{\dot{\mathfrak{g}}}. \tag{5.41}$$

Proposition 5.5. *(a) The operators* $L_n^{\mathfrak{g},\dot{\mathfrak{g}}}$ *commute with* $\tilde{\mathcal{L}}(\dot{\mathfrak{g}})$.

(b) The map $d_n \mapsto L_n^{\mathfrak{g},\dot{\mathfrak{g}}}$, $c \to c(k) - c(\dot{k})$ *defines a representation of* Vir *on* V.

Definition. *The* Vir*-module defined in Proposition 5.5 is called the* coset Vir-module.

The benefit of the coset construction is that it is a method of constructing unitary highest-weight representations of the Virasoro algebra Vir, and further allows the classification of unitary highest weight representations of the Virasoro algebra.

The Virasoro algebra is widely used in conformal field theory and string theory. It is the quantum version of the conformal algebra in two dimensions. The representation theory of strings on the worldsheet, the two-dimensional manifold which describes the embedding of a string in spacetime, reduces to the representation theory of the Virasoro algebra. The trace of the stress-energy tensor, the tensor that describes the density and flux of energy and momentum in spacetime, and which generalizes the stress tensor of Newtonian physics, vanishes in the constraints of the Virasoro algebra. In this way, the Virasoro algebra encodes two-dimensional conformal symmetry and is therefore a fundamental object in physics.

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