Grothendieck-Riemann-Roch

Abstract The Chern character does not commute with proper pushforward. In other words, let $f : X \to Y$ be a proper morphism of nonsingular varieties. Then the square

$$\begin{array}{ccc} K(X) & \xrightarrow{f_*} & K(Y) \\ \downarrow{\scriptstyle \mathrm{ch}_X} & & \downarrow{\scriptstyle \mathrm{ch}_Y} \\ A(X) \otimes_{\mathbf{Z}} \mathbf{Q} & \xrightarrow{f_*} & A(Y) \otimes_{\mathbf{Z}} \mathbf{Q} \end{array}$$

doesn't commute, where $A(X)$ denotes the Chow ring and ch is the Chern character. The Grothendieck-Riemann-Roch theorem states that

$$\mathrm{ch}(f_*\alpha) \cdot \mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha) \cdot \mathrm{td}(T_X)),$$

where td denotes Todd genus. We describe the proof when f is a projective morphism.

1 Statement of the theorem

Fix a field k. In this document the word 'scheme' will mean 'k-scheme of finite type.' Let X be a scheme. $K^\circ(X)$ denotes the Grothendieck group of vector bundles on X. $K_\circ(X)$ denotes the Grothendieck group of coherent sheaves on X. If X is quasiprojective nonsingular, the canonical homomorphism

$$K^\circ(X) \to K_\circ(X)$$

is an isomorphism. This is because the local rings of X are regular, and hence of global dimension equal to their finite Krull dimension, which is bounded above by the dimension of X. Therefore any coherent sheaf $\mathscr{F}$ on X admits a finite locally free resolution

$$0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to \mathscr{F} \to 0,$$

yielding an inverse of the above homomorphism which takes $[\mathscr{F}]$ to $\sum_{i=0}^{n}(-1)^i[E_i]$. So, when we are studying a nonsingular variety X, we can write $K(X)$ with no ambiguity. The notation $H^i(X,\mathscr{F})$ denotes the ith right derived functor of the global sections functor Γ on X with coefficients in the sheaf $\mathscr{F}$.

Let X,Y be schemes. For any morphism $f : Y \to X$ there is an induced homomorphism

$$f^* : K^\circ X \to K^\circ Y,$$

taking a vector bundle $[E]$ to $[f^*E]$ where $f^*E = Y \times_X E$ is the pullback bundle. For any proper morphism of schemes $f : X \to Y$ there is a homomorphism

$$f_* : K_\circ X \to K_\circ Y$$

which takes $[\mathscr{F}]$ to $\sum_{i\geq 0}(-1)^i[R^i f_*\mathscr{F}]$, where $R^i f_*\mathscr{F}$ denotes ith higher direct image. For the remainder of this document, X will denote a smooth quasiprojective algebraic variety.

We consider for the moment the situation when X is moreover a complex variety. Then, we have the usual resolution of the constant sheaf $\mathbf{Z}$ by the complex of singular cochains, and characteristic classes of vector bundles on X lying in $H^*(X,\mathbf{Z})$. The *Chern character* $\mathrm{ch}(E)$ of a vector bundle E on X is defined by the formula

$$\mathrm{ch}(E) = \sum_{i=1}^{r} \exp(\alpha_i).$$

Here α_i are Chern roots for E. When E has a filtration with line bundle quotients L_i, then $\alpha_i = c_1(L_i) \in H^2(X,\mathbf{Z})$. The *Todd class* $\mathrm{td}(E)$ of a vector bundle E is defined by the formula

$$\mathrm{td}(E) = \prod_{i=1}^{r} Q(\alpha_i), \qquad \text{where } Q(x) = \frac{x}{1-e^{-x}}.$$

Since Chern roots are additive on exact sequences of bundles, td is multiplicative and ch additive. Moreover, if E and E' are vector bundles, $\mathrm{ch}(E\otimes E') = \mathrm{ch}(E)\cdot\mathrm{ch}(E')$. Therefore, ch descends to a homomorphism

$$\mathrm{ch} : K(X) \to H^*(X,\mathbf{Z}) \otimes \mathbf{Q} \cong H^*(X,\mathbf{Q}).$$

Note that the image of ch is contained in even cohomology.

Let $f : X \to Y$ be a proper morphism of smooth quasiprojective complex varieties. Then the Grothendieck-Riemann-Roch theorem states that for $\alpha \in K(X)$,

$$\mathrm{ch}(f_*\alpha)\cdot\mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha)\cdot\mathrm{td}(T_X))$$

in the ring $H^*(Y,\mathbf{Q})$. The map f_* on cohomology can be described in the following way. The class $\mathrm{ch}(\alpha)\cdot\mathrm{td}(T_X) \in H^*(X,\mathbf{Q})$ can be represented by an algebraic cycle

W on X. This cycle admits a locally finite triangulation, i.e. such that a compact subset of X intersects only finitely many simplices. This triangulation defines the class of W in Borel-Moore homology of X, which is by definition the homology of the complex of locally finite singular chains. The functoriality of these chains for a proper map $f : X \to Y$ is evident, since if $C \subset Y$ is compact, $f^{-1}C$ is also, and hence only finitely many (singular) simplices have image in Y intersecting C. By assumption, X and Y are smooth quasiprojective complex varieties. Poincaré duality extends to give an isomorphism

$$H^i(X,\mathbf{Z}) \cong H^{\mathrm{BM}}_{2n-i}(X,\mathbf{Z}),$$

where X has algebraic dimension n and H^{BM}_* denotes Borel-Moore homology (likewise for Y). This defines the map f_*.

By taking the theorem in the special case of $f : X \to \{\cdot\}$, one recovers the theorem of Hirzebruch-Riemann-Roch (HRR), which in our case says, for E a vector bundle on a nonsingular complex projective variety X,

$$\chi(X,E) = \int_X \mathrm{ch}(E)\cdot \mathrm{td}(T_X).$$

Here, the notation $\int_X$ means to take the cohomology in the highest degree, represent it as a linear combination of points via Poincaré duality, and count these points with multiplicity. Let us recover from this the statement of classical Riemann-Roch, which applies when X is a complete nonsingular curve of genus g. The geometric genus of a curve is by definition $\dim_k H^0(X,\omega_X)$, the dimension of the global sections of the canonical sheaf $\omega_X = \Omega_{X/k}$ (our remarks so far restrict us in what follows to the case $k = \mathbf{C}$). The arithmetic genus of a curve is $\dim_k H^1(X,\mathscr{O}_X)$. It happens that ω_X is a dualizing sheaf on X, and by Serre duality the vector spaces $H^0(X,\omega_X)$ and $H^1(X,\mathscr{O}_X)$ are dual to one another. Their dimension can be taken as the definition of the genus of a (complete nonsingular) curve. In any event, since $H^0(X,\mathscr{O}_X) = k$, this, together with HRR and the computation of the first two terms of the Todd class of a line bundle

$$\frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}x^{2k},$$

reveals that

$$1-g = \chi(X,\mathscr{O}_X) = \frac{1}{2}\int_X c_1(T_X).$$

If E is a vector bundle of rank e on X, then since ch is additive on short exact sequences and $c_1(E)$ is simply the sum of the Chern roots of E, $\mathrm{ch}(E) = e + c_1(E)$, and we have

$$\chi(X,E) = \int_X c_1(E) + e(1-g).$$

In particular, when $E = \mathscr{O}(D)$ is a line bundle,

$$\chi(X, \mathscr{O}(D)) = \deg(D) + 1 - g.$$

The Chow ring Let X be a smooth scheme. The Chow group $A_k(X)$, resp. $A^k(X)$ denotes the group of algebraic cycles of dimension, resp. codimension, k on X modulo rational equivalence. We denote algebraic cycles of dimension (resp. codimension) k on X by Z_kX, resp. Z^kX. Since X is smooth, the intersection product gives $A^*(X)$ the structure of commutative, graded ring with unit $[X]$. The notation $A(X)_{\mathbf{Q}}$ denotes $A(X) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Characteristic classes can be defined very easily as operators on the Chow ring. When L is a line bundle on X, find D a Cartier divisor on X with $\mathscr{O}(D) \cong L$. Then $c_1(L) \frown \alpha = [D] \cdot \alpha$ for $\alpha \in A^*(X)$; i.e. the action of $c_1(L)$ on the Chow ring of X is simply intersection with D. The first Chern class of a bundle E of rank r can be defined simply in terms of determinants, as $c_1(E) = c_1(\wedge^r E)$. To define the higher classes, we mention the splitting construction.

Given a finite collection of vector bundles $\mathscr{S}$ of vector bundles on a scheme X, there is a flat morphism $f : X' \to X$ such that

1. $f^* : A(X) \to A(X')$ is injective, and
2. for each E in $\mathscr{S}$, f^*E has a filtration by subbundles

$$f^*E = E_r \supset E_{r-1} \supset \cdots \supset E_1 \supset E_0 = 0$$

with line bundle quotients $L_i = E_i/E_{i-1}$.

The flag varieties of vector bundles provide the desired X'.

Now, with f as in the splitting construction, f^*E is filtered with line bundle quotients L_i. Define the *Chern polynomial*

$$c_t(f^*E) = \prod_{i=1}^{r}(1 + c_1(L_i)t).$$

Then $c_i(f^*E)$ is simply the coefficient of t^i in $c_t(f^*E)$. By insisting that the $c_i(E)$ are natural under flat pullback, we determine the $c_i(E)$ completely.

We then define Chern character ch and Todd class td identically to as before. Defined algebraically in this way, the Chern character actually induces an isomorphism

$$\mathrm{ch} : K(X)_{\mathbf{Q}} \to A(X)_{\mathbf{Q}}$$

of $\mathbf{Q}$-algebras. To see this, one passes to associated graded groups, giving A_*X its natural filtration and $K_\circ X$ its topological filtration defined by letting $F_k K_\circ X$ be the subgroup generated by coherent sheaves whose support has dimension at most k. There is a surjection $A_kX \to \mathrm{Gr}_k K_\circ X$, which, composed with ch, gives the natural inclusion of A_*X in $A_*X_{\mathbf{Q}}$. Since, after tensoring with $\mathbf{Q}$, ch determines an isomorphisms on associated graded groups, the same must hold on the original groups.

The Grothendieck-Riemann-Roch theorem remains true if you replace ordinary cohomology with the Chow ring. Namely, for $\alpha \in K(X)$, $f : X \to Y$ a projective morphism of nonsingular schemes (over any field),

$$\mathrm{ch}(f_*\alpha) \cdot \mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha) \cdot \mathrm{td}(T_X))$$

in the ring $A^*(Y)$. Here f_* is the proper pushforward of algebraic cycles.

2 Proof of the theorem

Let X be a nonsingular scheme. The proof of the theorem is organized in the following way. First we consider the toy case of the zero-section imbedding of X in a vector bundle on it. After turning briefly to discuss the K-theory of a projective bundle on X, we discuss the deformation to the normal cone of a closed imbedding. In the final subsection, we use the results for the K-theory of a projective bundle to prove the main theorem in the case of a projection, and the deformation to the normal cone to prove the theorem in the case of a closed imbedding. Together, these constitute the proof of Grothendieck-Riemann-Roch in the case of a projective morphism.

2.1 The toy case

Let us first consider the special case of a closed imbedding $f : X \to Y$ where $Y = P(N \oplus 1)$ for N an arbitrary vector bundle of rank d on X; in particular, f is the zero section imbedding of X in N, followed by the canonical open imbedding of N in $P(N \oplus 1)$. Let p denote bundle projection $Y \to X$, and let Q be the universal quotient bundle, of rank d, on Y. Let s denote the section of Q determined by the projection of the trivial factor in $p^*(N \oplus 1)$ to Q. Then s is a regular section, and

$$f_*(f^*\alpha) = c_d(Q) \cdot \alpha. \tag{1}$$

Additionally, the Koszul complex

$$0 \to \wedge^d Q^\vee \to \ldots \to \wedge^2 Q^\vee \to Q^\vee \xrightarrow{s^\vee} \mathscr{O}_Y \to f_*\mathscr{O}_X \to 0$$

is a resolution of the sheaf $f_*\mathscr{O}_X$. For any vector bundle E on X, we therefore have the explicit resolution of E

$$0 \to \wedge^d Q^\vee \otimes p^*E \to \ldots \to Q^\vee \otimes p^*E \to p^*E \to f_*E \to 0.$$

Hence,

$$\mathrm{ch}\, f_*[E] = \sum_{p=0}^{d} (-1)^p \,\mathrm{ch}(\wedge^p Q^\vee) \cdot \mathrm{ch}(p^*E). \tag{2}$$

Chern character ch and Todd class td are related by the formula

$$\sum_{p=0}^{d}(-1)^p \operatorname{ch}(\wedge^d Q^\vee) = c_d(Q)\cdot \operatorname{td}(Q)^{-1}. \tag{3}$$

Combining (1), (2), and (3), we write

$$\operatorname{ch} f_*E = c_d(Q)\operatorname{td}(Q)^{-1}\cdot \operatorname{ch}(p^*E) = f_*(f^*\operatorname{td}(Q)^{-1}\cdot f^*\operatorname{ch}(p^*E)).$$

Since $f^*Q = N$ and $f^*p^*E = E$, this can be rewritten as

$$\operatorname{ch} f_*E = f_*(\operatorname{td}(N)^{-1}\cdot \operatorname{ch}(E)). \tag{4}$$

By the multiplicativity of td and the exact sequence of vector bundles arising from a regular imbedding of a nonsingular subvariety in a nonsingular variety

$$0 \to T_X \to f^*T_Y \to N_X Y \to 0,$$

we find

$$\operatorname{td}(N)^{-1} = f^*\operatorname{td}(T_Y)^{-1}\cdot \operatorname{td}(T_X).$$

The right side of (4) is therefore

$$f_*(f^*\operatorname{td}(T_Y)^{-1}\cdot \operatorname{td}(T_X)\cdot \operatorname{ch} E) = \operatorname{td}(T_Y)^{-1}\cdot f_*(\operatorname{td}(T_X)\cdot \operatorname{ch} E),$$

and (4) can be rewritten as

$$\operatorname{ch}(f_*E)\cdot \operatorname{td}(T_Y) = f_*(\operatorname{ch}(E)\cdot \operatorname{td}(T_X)). \tag{5}$$

2.2 $K(\mathbf{P})$

Theorem 1. *Let X be a nonsingular scheme, E a vector bundle on X of rank $n+1$, $q:\mathbf{P}=\mathbf{P}(E)\to X$ the projection. Then, $K(\mathbf{P})$ is a free $K(X)$-module generated by the classes of $\mathscr{O}(-i)$, $i=0,\dots,n$.*

Proof (of Theorem). There are two steps: first, showing that the classes of $\mathscr{O}(-i)$, $i=0,\dots,n$ generate a free submodule of $K(\mathbf{P})$ over $K(X)$; second, showing that these classes generate $K(\mathbf{P})$ as a module over $K(X)$. The below commutative diagram establishes notation.

$$\begin{tikzcd}
 & \mathbf{P} \arrow[d, "\Delta"] \arrow[ddl, bend right, "\mathrm{id}"'] \arrow[ddr, bend left, "\mathrm{id}"] & \\
 & \mathbf{P}\times_X \mathbf{P} \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
 \mathbf{P} \arrow[dr, "q"] & & \mathbf{P} \arrow[dl, "q"] \\
 & X &
\end{tikzcd}$$

For the first step, it suffices to write down projection maps $K(\mathbf{P}) \to K(X)$. For $i = 0, 1, \ldots, n$,

$$R^a q_*(\Omega^j_{\mathbf{P}/X}(j-i)) = \begin{cases} \mathscr{O}_X & \text{if } a = i = j \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $H = \Omega^1_{\mathbf{P}/X}(1)$, and $e_i : K(\mathbf{P}) \to K(X)$ is given by

$$e_i(?) = (-1)^i q_*(? \otimes \wedge^i H),$$

then e_i assumes the value $[\mathscr{O}_X]$ on $[\mathscr{O}(-i)]$ and 0 on $[\mathscr{O}(-j)]$ for $0 \leq j \neq i \leq n$. Hence the classes of $\mathscr{O}(-i)$, $i = 0, \ldots, n$ generate a free module over $K(X)$.

For the second step, we must show that every coherent sheaf on projective space is equal to a linear combination of the $\mathscr{O}(-i)$, $i = 0, \ldots, n$, in $K(\mathbf{P})$. The Koszul complex

$$0 \to \mathscr{O}(-n) \boxtimes \wedge^n H \to \cdots \to \mathscr{O}(-2) \boxtimes \wedge^2 H \to \mathscr{O}(1) \boxtimes H \to \mathscr{O}_{\mathbf{P}\times\mathbf{P}} \to \mathscr{O}_\Delta \to 0$$

is in fact a resolution of the diagonal $\mathscr{O}_\Delta = \Delta_* \mathscr{O}_{\mathbf{P}} \subset \mathbf{P} \times_X \mathbf{P}$ for projective space. Therefore, for a coherent sheaf ? on $\mathbf{P}$,

$$\begin{aligned}
? &= p_{1*}(\mathscr{O}_\Delta \otimes p_2^*(?)) \\
&= p_{1*}\left(\sum_{i=0}^n (-1)^i (\mathscr{O}(-i) \boxtimes \wedge^i H) \otimes p_2^*(?)\right) \\
&= p_{1*}\left(\sum_{i=0}^n (-1)^i \mathscr{O}(-i) \boxtimes (\wedge^i H \otimes ?)\right) \\
&= \sum_{i,j=0}^n (-1)^{i+j} \mathscr{O}(-i) \otimes_{\mathscr{O}_X} R^j q_*(\mathbf{P}, \wedge^i H \otimes ?)
\end{aligned}$$

in $K(\mathbf{P})$, where we have written simply ?, etc. for the class [?] in $K(\mathbf{P})$, and the last equality is by Künneth. This proves step 2, and the theorem.

2.3 Deformation to the normal cone

Let X be a closed subscheme of Y. The claim is that there is a scheme $M = M_X Y$, a closed imbedding $X \times \mathbf{P}^1 \hookrightarrow M$, and a flat morphism $\rho : M \to \mathbf{P}^1$ so that

$$\begin{tikzcd} X \times \mathbf{P}^1 \arrow[rr, hook] \arrow[dr, "\mathrm{pr}"'] & & M \arrow[dl, "\rho"] \\ & \mathbf{P}^1 & \end{tikzcd}$$

commutes, and such that

1. Over $\mathbf{P}^1 - \{\infty\} = \mathbf{A}^1$, $\rho^{-1}(\mathbf{A}^1) = Y \times \mathbf{A}^1$ and the imbedding is the trivial one
$$X \times \mathbf{A}^1 \hookrightarrow Y \times \mathbf{A}^1.$$
2. Over ∞, the divisor $M_\infty = \rho^{-1}(\infty)$ is the sum of two effective divisors
$$M_\infty = P(C \oplus 1) + \tilde{Y}$$
where $\tilde{Y}$ is the blowup of Y along X. The imbedding of $X = X \times \{\infty\}$ in M_∞ is the zero-section imbedding of X in C followed by the canonical open imbedding of C in $P(C \oplus 1)$. The divisors $P(C \oplus 1)$ and $\tilde{Y}$ intersect in the scheme $P(C)$, which is imbedded as the hyperplane at infinity in $P(C \oplus 1)$, and as the exceptional divisor in $\tilde{Y}$. In particular, the image of X in M_∞ is disjoint from $\tilde{Y}$. Letting $M^\circ = M_X^\circ Y$ be the complement of $\tilde{Y}$ in M, one has a family of imbeddings of X:
$$\begin{tikzcd} X \times \mathbf{P}^1 \arrow[rr, hook] \arrow[dr, "\mathrm{pr}"'] & & M^\circ \arrow[dl, "\rho^\circ"] \\ & \mathbf{P}^1 & \end{tikzcd}$$
which deforms the given imbedding of X in Y to the zero-section imbedding of X in C.

Such an M is found by blowing up $Y \times \mathbf{P}^1$ along $X \times \{\infty\}$.

2.4 Proof of Riemann-Roch for a projective morphism

Theorem 2. *Let $f : X \to Y$ be a projective morphism of nonsingular varieties. Then for all $\alpha \in K(X)$,*
$$\mathrm{ch}(f_*\alpha) \cdot \mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha) \cdot \mathrm{td}(T_X))$$
in $A(Y)_{\mathbf{Q}}$.

Let

$$\tau_X : K(X) \to A(X)_{\mathbf{Q}}$$

be defined by

$$\tau_X(\alpha) = \mathrm{ch}(\alpha) \cdot \mathrm{td}(T_X).$$

Then the theorem can be reformulated as 'τ commutes with pushforward under a projective morphism'; i.e. $f_* \circ \tau_X = \tau_Y \circ f_*$. It follows that If the theorem is valid for a closed imbedding $g : X \to Y \times \mathbf{P}^m$ and for the projection $p : Y \times \mathbf{P}^m \to Y$, then it is valid for the projective morphism $g \circ p$.

Riemann-Roch for closed imbeddings The name of the game is to reduce the case of $f : X \to Y$ a closed imbedding to the toy case. Let N denote the normal bundle to X in Y. We shall use the deformation to the normal bundle to deform the imbedding f into the imbedding $\tilde{f} : X \to P(N \oplus 1)$ discussed at the beginning of this section. We have a diagram

$$\begin{tikzcd}
X \arrow[r, "\overline{f}"] \arrow[d, "i_\infty"] & P(N \oplus 1) + \arrow[dr, "k"] & \tilde{Y} \arrow[d, "l"] & = M_\infty \arrow[r] \arrow[dl, "j_\infty"] & \{\infty\} \arrow[d] \\
X \times \mathbf{P}^1 \arrow[rr, "F"] & & M \arrow[rr] & & \mathbf{P}^1 \\
X \arrow[u, "i_0"] \arrow[rr, "f"] & & Y \arrow[u, "j_0"] & = M_0 \arrow[r] & \{0\} \arrow[u]
\end{tikzcd}$$

where M is the blowup of $Y \times \mathbf{P}^1$ along $X \times \{\infty\}$. We may assume $\alpha = [E]$, with E a vector bundle on X. Let $\tilde{E} = p^* E$, where p is the projection from $X \times \mathbf{P}^1$ to X. Choose a resolution $G.$ of $F_*(\tilde{E})$ on M:

$$0 \to G_n \to G_{n-1} \to \ldots \to G_0 \to F_*(\tilde{E}) \to 0. \tag{*}$$

Since $X \times \mathbf{P}^1$ and M are both flat over $\mathbf{P}^1$, the restrictions of the sequence (*) to the fibers M_0 and M_∞ remains exact. Therefore $j_0^* G.$ resolves $j_0^*(F_*(\tilde{E}))$ and $j_\infty^* G.$ resolves $j_\infty^*(F_*(\tilde{E}))$. Since $j_0^* F_* \tilde{E} = f_* i_0^* \tilde{E} = f_*(E)$,

(i) $j_0^* G.$ resolves $f_*(E)$ on $Y = M_0$.

Similarly, $j_\infty^* G.$ resolves $\overline{f}_*(E)$ on M_∞. But, $\tilde{f}(X)$ is disjoint from $\tilde{Y}$. Therefore

(ii) $k^* G.$ resolves $\tilde{f}_*(E)$ on $P(N \oplus 1)$, and
(iii) $l^* G.$ is acyclic.

For a complex $F.$ of vector bundles, we write $\mathrm{ch}(F.)$ for the alternating sum $\sum (-1)^i \mathrm{ch}(F_i)$. We compute the image of $\mathrm{ch}(f_* E)$ in $A(M)_{\mathbf{Q}}$ (writing $\mathrm{ch}(f_* E)$ in lieu of $\mathrm{ch}(f_* E) \cdot [Y]$):

$$\begin{aligned}
j_{0*}(\mathrm{ch}(f_* E)) &= j_{0*}(\mathrm{ch}(j_0^* G.)) && \text{by (i)} \\
&= \mathrm{ch}(G.) \cdot j_{0*}[Y] && \text{(projection formula for Chern classes)} \\
&= \mathrm{ch}(G.) \cdot (k_*[P(N \oplus 1)] + l_*[\tilde{Y}]
\end{aligned}$$

(by the basic fact that $[M_0]-[M_\infty]=[\operatorname{div}\rho]=0$ in $A(M)_{\mathbf{Q}}$)

$$\begin{aligned}
&= k_*(\mathrm{ch}(k^*G.)) + l_*(ch(l^*G.)) && \text{(projection formula)}\\
&= k_*(\mathrm{ch}(\overline{f}_*E)) + 0 && \text{by (ii) and (iii).}
\end{aligned}$$

The morphism $\overline{f}$ was precisely the object of study at the beginning of this section. Equation (4) of that section allows us to write

(iv) $j_0 * \mathrm{ch}(\overline{f}_*E)) = k_*(\overline{f}_*(\mathrm{td}(N)^{-1}\cdot \mathrm{ch}(E)))$ in $A(M)_{\mathbf{Q}}$.

Let $q : M \to Y$ be the composite of the blowdown $M \to Y\times \mathbf{P}^1$ followed by the projection. By construction of M, $q\circ j_0 = \mathrm{id}_Y$, and $q\circ k\circ \overline{f} = f$. Applying q_* to (iv), we find

$$\mathrm{ch}(f_*E) = f_*(\mathrm{td}(N)^{-1}\cdot \mathrm{ch}(E)).$$

The theorem now follows from the same manipulations as were used to pass from (4) to (5) in the toy case.

Riemann-Roch for the projection Consider first more generally the projection $f : Y\times Z \to Y$, with Z nonsingular. There is a commutative diagram

$$\begin{array}{ccc}
K(Y)\otimes K(Z) & \xrightarrow{\tau_Y\otimes\tau_Z} & A(Y)_{\mathbf{Q}}\otimes A(Z)_{\mathbf{Q}}\\
\big\downarrow \times & & \big\downarrow \times\\
K(Y\times Z) & \xrightarrow{\tau_{Y\times Z}} & A(Y\times Z)_{\mathbf{Q}}.
\end{array}$$

Since the Todd class is multiplicative, $\mathrm{td}(T_{Y\times Z}) = \mathrm{td}(T_Y)\times \mathrm{td}(T_Z)$. If $Z=\mathbf{P}^m$, the left vertical map is surjective, and $K(\mathbf{P}^m)$ is generated by $[\mathscr{O}(-i)]$, $i=0,1,\ldots,m$, both statements following from Theorem 1. It suffices therefore to verify the theorem for the projection from $\mathbf{P}^m$ to a point and $\alpha = [\mathscr{O}(-i)]$; i.e. to verify the formula

$$\int_{\mathbf{P}^m} \mathrm{ch}(\mathscr{O}(-i))\cdot \mathrm{td}(T_{\mathbf{P}^m}) = \chi(\mathbf{P}^m, \mathscr{O}(-i)).$$

Here, if $p : \mathbf{P}^m \to \operatorname{Spec} k$ is the projection, the notation $\int_{\mathbf{P}^m}$ denotes the extension of the proper pushforward $p_* : A_0(\mathbf{P}^m) \to A_0(\operatorname{Spec} k)$ by zero to the whole Chow ring $A(\mathbf{P}^m)$. As both ch and χ are homomorphisms of rings, in particular it suffices to verify the formula after flipping sign

$$\int_{\mathbf{P}^m} \mathrm{ch}(\mathscr{O}(n))\cdot \mathrm{td}(T_{\mathbf{P}^m}) = \chi(\mathbf{P}^m, \mathscr{O}(n)),$$

$n=0,1,\ldots,m$.

Now, $\mathrm{td}(T_{\mathbf{P}^m}) = (x/1-e^{-x})^{m+1}$, where $x = c_1(\mathscr{O}_{\mathbf{P}^m}(1))$, and compute

$$\int_{\mathbf{P}^m} e^{nx}x^{m+1}/(1-e^{-x})^{m+1} = \binom{n+m}{n}.$$

To see this, note that the integrand is a power series in x, for which we want the coefficient of x^m. Dividing the integrand by x^{m+1}, this is the same as computing the residue of $e^{nx}/(1-e^{-x})^{m+1}$. Changing variables $y = 1-e^{-x}$ this is the same as asking for the residue of $(1-y)^{-n-1}y^{-m-1}$, or the coefficient of the term of degree m in $(1-y)^{-n-1} = (1+y+y^2+\cdots)^{n+1}$, which is $\binom{n+m}{n}$. On the other hand, the sheaves $\mathscr{O}(n)$ for $n = 0,1,\ldots,m$ are generated by global sections and have no higher cohomology; hence

$$\chi(\mathbf{P}^m, \mathscr{O}(n)) = \dim_k \operatorname{Sym}^n k^{m+1} = \binom{n+m}{m}.$$

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