Research Statement

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1 Introduction

I have been working under the supervision of David Nadler in the fields of symplectic geometry, combinatorics, and representation theory. Recently, mathematicians have been interested in a collection of powerful invariants associated to symplectic manifolds, in particular the Fukaya category and related ideas. These invariants are at the heart of deep conjectures in mathematics and theoretical physics, for example mirror symmetry.

My research is focused on a combinatorial approach to understanding symplectic manifolds and the associated invariants. Given a Lagrangian skeleton of a symplectic manifold (or, a Legendrian skeleton of a contact manifold), one can study properties of the ambient manifold via microlocal sheaf theory applied to the singular space- see for example [9], [2]. In [6] Nadler introduced a collection of singular stratified spaces, termed arboreal singularities, which in a sense (discussed more thoroughly in [8]) provide the basic combinatorial building blocks from which one can create a sufficiently general Lagrangian skeleton. This yields an essentially combinatorial model for calculating categories of microlocal sheaves on symplectic manifolds.

In line with this philosophy, my research centers on singular spaces which have arboreal singularities (henceforth called "arboreal spaces"), equipped with some additional local combinatorial data called a "cyclic structure". In the simplest case, a one-dimensional arboreal space is a trivalent graph, that is, a graph in which each vertex has three edges connected to it. A cyclic structure is the choice of a cyclic order on the three edges adjacent to each vertex. Such objects are called **ribbon graphs** and have been studied extensively: It is known that ribbon graphs label cells in a cell-decomposition of the moduli space of punctured Riemann surfaces ([3]. [5]). Furthermore, in [10], Sibillia, Treumann and Zaslow construct a sheaf of dg-categories associated to a ribbon graph meant to model the Fukaya category of the surface. My research can be thought of as an extention of the theory of ribbon graphs to higher dimensions.

The central accomplishments of my research so far have been the definition of cyclic structure and observing that it defines (up to quasi-equivalence) a sheaf of dg-categories on an arboreal singularity (see Sections 2 and 3). I have also worked to understand combinatorial 'moves' or 'mutations' of cyclic arboreal spaces (Section 4). My main long-term goals revolve around understanding whether, like ribbon graphs, cyclic arboreal spaces determine the structure of higher-dimensional symplectic manifolds and can be used to compute the Fukaya category (Section 5).

2 Cyclic Structures

My work thus far has been focused on the purely combinatorial setting; that is, when the topological spaces being considered are finite posets with the upwards-closed topology. As explained in [6], to a tree T one can construct a poset $\mathcal{P}_T$, whose elements are labelled by **correspondences:** diagrams of the form:

$$(R \twoheadleftarrow S \hookrightarrow T)$$

Where the left-hand arrow is a quotient map and the right map an embedding of trees. For example, if one takes $T = A_n$, the tree with n vertices connected in a line, one finds $\mathcal{P}_T$ to be the cone over the $n-2-$skeleton of the $n-$simplex.

If one adds orientations to the edges of T to obtain a directed tree $\vec{T}$, one obtains the poset $\mathcal{P}_{\vec{T}}$, whose elements are labelled by orientation-preserving correspondences. This poset is canonically isomorphic to $\mathcal{P}_T$, but the extra structure allows us to construct a sheaf of dg-categories on $\mathcal{P}_{\vec{T}}$, whose stalk at a point labelled by the correspondence $(\vec{R} \twoheadleftarrow \vec{S} \hookrightarrow \vec{T})$ is the dg derived category $\mathrm{Rep}(\vec{R})$ of finite-dimensional $k[\vec{R}]-$modules over some fixed field k. Let $\mathcal{Q}_{\vec{T}}$ denote this sheaf.

Given any poset $\mathcal{P}$, a poset isomorphism $\varphi : \mathcal{P} \xrightarrow{\sim} \mathcal{P}_{\vec{T}}$ gives us a sheaf $\varphi^*\mathcal{Q}_{\vec{T}}$ of $dg-$categories on $\mathcal{P}$. My first result addresses the question of when two isomorphisms $\varphi : \mathcal{P} \xrightarrow{\sim} \mathcal{P}_{\vec{T}}$, $\varphi' : \mathcal{P} \xrightarrow{\sim} \mathcal{P}_{\vec{T}'}$ give rise to quasi-equivalent sheaves of dg-categories. Equivalently, when does a poset isomorphism $\varphi : \mathcal{P}_{\vec{T}} \xrightarrow{\sim} \mathcal{P}_{\vec{T}'}$ satisfy $\varphi^*\mathcal{Q}_{\vec{T}'} \cong \mathcal{Q}_{\vec{T}}$?

The answer is given by the notion of a **cyclic structure**. Let $\mathcal{P}$ be a combinatorial arboreal singularity- that is, a poset which is isomorphic to $\mathcal{P}_T$ for some T (henceforth a "C-Arb singularity"). For $x \in \mathcal{P}$, we say x has **codimension-one** if the only elements greater than it are maximal. For C-Arb singularities $\mathcal{P}$, with $x \in \mathcal{P}$ having codimension-one, x will have exactly 3 elements greater than it.

Definition 2.1. A **pre-cyclic structure** $\mathcal{O}$ on $\mathcal{P}$ is a choice, for all codimension-one elements $x \in \mathcal{P}$, a cyclic order $\mathcal{O}^x$ on the three elements greater than x.

For any directed tree $\vec{T}$ there is an associated pre-cyclic structure $\mathcal{O}_{\vec{T}}$ on $\mathcal{P}_{\vec{T}}$. My first result, to appear in [11], is the following theorem:

Theorem 2.2. *If $\varphi : \mathcal{P}_{\vec{T}} \xrightarrow{\sim} \mathcal{P}_{\vec{T}'}$ is an isomorphism, then φ preserves the sheaf of categories if and only if it preserves the pre-cyclic structure:*

$$\varphi^*\mathcal{Q}_{\vec{T}'} \cong \mathcal{Q}_{\vec{T}} \text{ if and only if } \varphi^*\mathcal{O}_{\vec{T}'} = \mathcal{O}_{\vec{T}}$$

This result immediately suggests the following definition:

Definition 2.3. A pre-cyclic structure $\mathcal{O}$ on $\mathcal{P}$ is a **cyclic structure** iff there exists an isomorphism $\varphi : \mathcal{P} \xrightarrow{\sim} \mathcal{P}_{\vec{T}}$ such that $\mathcal{O} = \varphi^*\mathcal{O}_{\vec{T}}$

Theorem 2.2 implies that a cyclic structure on a C-Arb singularity $\mathcal{P}$ determines, up to quasi-equivalence, a sheaf of $dg-$categories on $\mathcal{P}$. However, Definition 2.3 does little to illuminate what distinguishes cyclic structures among the pre-cyclic structures. In certain cases, cyclic structures admit very nice descriptions. As an example, consider $\mathcal{P}_{A_n}$. As explained above, we have the following description of $\mathcal{P}_{A_n}$, proven in [6]:

Lemma 2.4. *$\mathcal{P}_{A_n}$ is the cone over the $n-2-$skeleton of the $n-$simplex. Explicitly, if S denotes a set of $n+1$ elements, for $n \geq 1$, we have $\mathcal{P}_{A_n} \cong \{X \subseteq S \mid |X| \leq n-1\}$.*

Then in [11] I prove:

Proposition 2.5. *Fix an isomorphism $\mathcal{P}_{A_n} \cong \{X \subseteq S \mid |X| \leq n-1\}$. Then cyclic structures on $\mathcal{P}_{A_n}$ are in natural bijection with cyclic orders on S.*

We see that there is a cyclic symmetry here. In [7], Nadler gives a presentation of $\mathrm{Rep}(\vec{A}_n)$ as an $A_\infty-$category in which the cyclic symmetry is manifest. It is a goal of future research to have a similarly ‘symmetric’ description for arbitrary arboreal singularities.

In general, cyclic structures have a nice description in terms of local conditions. In my thesis, a proof of the following theorem will appear:

Theorem 2.6. *A pre-cyclic structure on $\mathcal{P}$ is a cyclic structure if and only if for any open subset U isomorphic to $\mathcal{P}_{A_3}$ or $\mathcal{P}_{S_4}$, the restriction of the pre-cyclic structure to U is a cyclic structure.*

(In the above theorem, S_{n+1} refers to a “star”: the tree with n vertices connected a central vertex.)

3 Reflection Functors

An important step in the proof of Theorem 2.2 is the notion of a ‘sheafy’ reflection functor. Let $\vec{T}$ denote a directed tree, $V(\vec{T})$ the set of vertices, and $v \in V(\vec{T})$ a *source* (all edges incident to v point away from v) or a *sink* (toward v). Let $r_v\vec{T}$ denote the tree with the arrows incident to v reversed. Then it is well-known (see [1]) that one has a quasi-equivalence of dg-categories $\mathcal{F}_v^{\pm} : \mathrm{Rep}(\vec{T}) \xrightarrow{\sim} \mathrm{Rep}(r_v\vec{T})$, where the $\mathcal{F}_v^{\pm}$ are called *reflection functors* (The sign depends on whether v is a source or sink).

In my work, I extend this reflection to act on the entire diagram of (directed) correspondences. Specifically, for $v \in V(\vec{T})$ of degree ≤ 2, I can associate to any correspondence $\mathfrak{p} = (\vec{R} \twoheadleftarrow \vec{S} \hookrightarrow \vec{T})$ a new one $r_v\mathfrak{p} = (\vec{R}' \twoheadleftarrow \vec{S}' \hookrightarrow r_v\vec{T})$. Then I construct a quasi-equivalence $\mathrm{Rep}(\vec{R}) \xrightarrow{\sim} \mathrm{Rep}(\vec{R}')$, compatible with the restrictions. There are two cases (depending on $\mathfrak{p}$), in one case $\vec{R}' = r_w\vec{R}$ for some $w \in V(\vec{R})$, and in the other $\vec{R}' = \vec{R}$. In either case, one has diagrams:

$$\begin{tikzcd} \mathrm{Rep}(\vec{T}) \arrow[r, "\mathcal{F}_v^{\pm}"] \arrow[d, "\rho_{\mathfrak{p}}"'] & \mathrm{Rep}(r_v\vec{T}) \arrow[d, "\rho_{r_v\mathfrak{p}}"] \\ \mathrm{Rep}(\vec{R}) \arrow[r, "\mathcal{F}_w^{\pm}"] & \mathrm{Rep}(r_w\vec{R}) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Rep}(\vec{T}) \arrow[r, "\mathcal{F}_v^{\pm}"] \arrow[d, "\rho_{\mathfrak{p}}"'] & \mathrm{Rep}(r_v\vec{T}) \arrow[d, "\rho_{r_v\mathfrak{p}}"] \\ \mathrm{Rep}(\vec{R}) \arrow[r, "id_{\mathrm{Rep}(\vec{R})}"] & \mathrm{Rep}(\vec{R}) \end{tikzcd}$$

Where these diagrams commute (up to a coherent collection of natural equivalences). This gives us an isomorphism $\varphi : \mathcal{P}_{\vec{T}} \to \mathcal{P}_{r_v\vec{T}}$ satisfying $\varphi^*\mathcal{O}_{r_v\vec{T}} = \mathcal{O}_{\vec{T}}$, and a quasi-equivalence $\varphi^*\mathcal{Q}_{r_v\vec{T}} \cong \mathcal{Q}_{\vec{T}}$. The proof of ‘if’ in Theorem 2.2 now essentially follows from a combinatorial argument contained in [11].

It is an interesting question for future research whether these ideas apply to more general mutations, for example, mutations for quivers with potential discussed in [4].

4 Combinatorial Moves

In the theory of trivalent ribbon graphs, there is a ‘move’ or that preserves the associated Riemann surface. It sometimes suggestively called the “H-to-I” move, Mulase and Penkava call it the “Fusion Move” in [5], and it is closely related to the “square move” and quiver mutations: (Figure 1).

In [5], it is explained that a ribbon graph Γ can be embedded into a punctured Riemann surface X_Γ such that the surface retracts onto Γ. The genus g and number of punctures n of X_Γ can be comptued

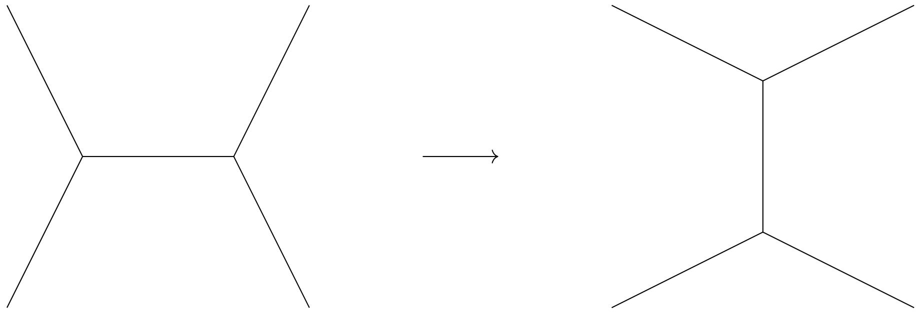

Figure 1: The H-to-I move

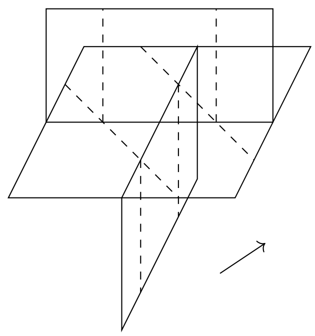

Figure 2: Total Space of the H-to-I move

combinatorially from the ribbon graph. Importantly, the square move preserves these invariants.

My next results concern generalizations of the H-to-I move to other C-Arb spaces, that is, describing mutations of C-Arb spaces which preserve certain local invariants (i.e. a dg-category). An important observation is that if one views the H-to-I move as a continuous deformation, the total space looks like an A_3 singularity (Figure 2).

We use this idea to formalize the notion of a (combinatorial) arboreal move. In the following definition, 'cyclic C-Arb Space' refers to a pair $(\mathcal{P}, \mathcal{O})$ such that for $x \in \mathcal{P}$, $N(x)$ is a C-Arb singularity, and $\mathcal{O}$ is a cyclic structure on $\mathcal{P}$.

Definition 4.1. For a tree T, an **arboreal move of type** T between cyclic C-Arb spaces $(\mathcal{P}_1\mathcal{O}_1)$ and $(\mathcal{P}_2, \mathcal{O}_2)$ consists of the following:

- A *total space* $(\tilde{\mathcal{P}}, \tilde{\mathcal{O}})$, which is a cyclic C-Arb singularity isomorphic to $(\mathcal{P}_{\vec{T}}, \mathcal{O}_{\vec{T}})$. ($\vec{T}$ being an orientation of T). So we have (up to quasi-equivalence) a sheaf $\mathcal{Q}$ of dg categories on $\tilde{\mathcal{P}}$.
- Open embeddings $\phi_i : \mathcal{P}_i \hookrightarrow \tilde{\mathcal{P}}$ with $\phi_i^* \tilde{\mathcal{O}} = \mathcal{O}_i$
- Let U_i denote the image of ϕ_i, and $\hat{0}$ the unique minimal element in $\tilde{\mathcal{P}}$. Then $U_1 \cup U_2 = \tilde{\mathcal{P}} \backslash \{\hat{0}\}$, and for any $x \in U_1 \cap U_2$, the closure $\mathrm{cl}(\{x\})$ is not contained in U_1 or in U_2.
- **Preservation of Categories:** The restrictions $\mathcal{Q}(\tilde{\mathcal{P}}) \to \mathcal{Q}(U_i)$ are quasi-equivalences of dg-categories.

The H-to-I move is an example of an arboreal move of type A_2.

Figure 3: Good Decompositions for $\mathcal{P}_{A_3}$ and $\mathcal{P}_{A_4}$

Such moves can be described using the following recipe: Write the total space as $\mathcal{P}_{\vec{T}}$. As in Definition 4.2, let $\hat{0}$ denote the minimal element of $\mathcal{P}_{\vec{T}}$, and E denote the set of elements x just greater than $\hat{0}$, i.e. such that $y < x \Leftrightarrow y = \hat{0}$. Consider a decomposition $E = E_1 \sqcup E_2$ and let U_{E_i} denote the smallest open neighborhood which contains each element of E_i for $i = 1, 2$. As long as the restrictions $\mathcal{Q}_{\vec{T}}(\mathcal{P}_{\vec{T}}) \to \mathcal{Q}_{\vec{T}}(U_i)$ are quasi-equivalences, this determines an arboreal move between U_{E_1} and U_{E_2}. It is not hard to show any move arises from such a construction.

Hence the question is: For which subsets $F \subset E$ is the restriction $\mathcal{Q}_{\vec{T}}(\mathcal{P}_{\vec{T}}) \to \mathcal{Q}_{\vec{T}}(U_F)$ a quasi-equivalence? Call such subsets **good**. Then moves are determined by good decompositions: decompositions $E = E_1 \sqcup E_2$ in which each E_i is good.

The answer for $\mathcal{P}_{A_n}$ is contained in the following theorem. Recall that giving a cyclic structure on $\mathcal{P}_{A_n}$ is equivalent to giving a cyclic order on E. Then:

Theorem 4.2. *$F \subseteq E$ is good if and only if it is not an interval with respect to the cyclic order.*

For example, for $\mathcal{P}_{A_3}$, $|E| = 4$ and this means E_1, E_2 must consist of two opposing points. This decomposition gives the H-to-I move. There is also only one move (up to equivalence) of type A_4. For $\mathcal{P}_{A_4}$, $|E| = 5$ and one of the E_i consists of two non-adjacent points, and the other consists of three. This is illustrated in Figure 4, where the decomposition of the cyclic set is illustrated by the shading of the dots.

5 Looking Forward: Goals of a Geometric Theory

Thus far, I have been primarily concerned with the combinatorics of arboreal posets, but it is natural to examine *topological arboreal spaces*, that is, stratified topological spaces whose strata poset is locally an arboreal singularity. In [6], Nadler explains how to construct a stratified space L_T whose strata poset is $\mathcal{P}_T$. The notion of pre-cyclic/cyclic structures on an arboreal space are the same as in the combinatorial realm.

An embedding of an arboreal space Γ into a symplectic manifold M as a stratified Lagrangian submanifold induces a pre-cyclic structure on Γ. Explicitly, any codimension-one cell in the skeleton looks locally like $\mathbb{R}^{n-1} \times L_{A_2}$, where L_{A_2} is a trivalent vertex- in other words, it looks like a $\mathbb{R}^{n-1}-$"spine" with three $\mathbb{R}^n$ "leaves". The symplectic structure induces a cyclic order on these leaves.

Given an arboreal space Γ, a **good** embedding of Γ as the skeleton of a symplectic manifold M is one in which the induced pre-cyclic structure is a cyclic structure. Figure 5 illustrates examples of good and not good embeddings of L_{A_3} into $T^*\mathbb{R}^2$ as the zero section plus the conormal to cooriented curves. A central goal is to show that, for a good embedding of Γ, symplectic invariants including the Fukaya category on M can be computed directly on Γ. Even stronger, I would hope that the cyclic structure *determines* the embedding, up to a natural notion of equivalence.

The last piece of the puzzle is incorporating moves. Arboreal moves can be extended to the geometric setting- it would be very interesting to show that mutations preserve the ambient symplectic manifold M.

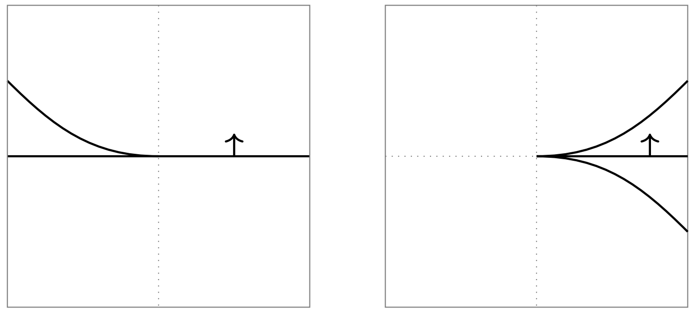

Figure 4: A good embedding (left) and not good embedding (right) of L_{A_3} into $T^*\mathbb{R}^2$.

Furthermore, perhaps all arboreal skeleta of a symplectic manifold M can be related by sequences of these mutations. Answering these questions would lead to a powerful combinatorial tool for understanding deep invariants of symplectic geometry.

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