Discussion Week 5: 2/23
MATH 110
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1. Let V and W be finite-dimensional vector spaces and $T : V \to W$ be a linear transformation. If $\beta = \{v_1, \ldots, v_n\}$ is a basis for V and $\gamma = \{w_1, \ldots, w_m\}$ is a basis for W:
   -What is the size of $[T]_\beta^\gamma$?
   -Write a formula for the jth column of $[T]_\beta^\gamma$

2. Define the linear transformation $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T(f(x)) = f'(x)$. Let $\beta = \{1+x, 1+x^2, x+x^2\}$. Find $[T]_\beta$ and $[T^2]_\beta$.

3. Let V, W, Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.
   (a) Prove that if UT is one-to-one and T is onto, then U is one-to-one.
   (b) Prove that if UT is invertible and U is one-to-one, then U and T are invertible.

4. If V and W are vector spaces, what does it mean to say V is **isomorphic** to W? If V and W are finite-dimensional, when are V and W isomorphic?

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Solutions:

1. $[T]_\beta^\gamma$ is $m \times n$. The jth column of $[T]_\beta^\gamma$ is $[T(v_j)]_\gamma$.

2.

$$T(b_1) = T(1+x) = 1$$

To find the β coordinate vector of 1, we need to solve:

$$1 = a(1+x) + b(1+x^2) + c(x+x^2) = (a+b) + (a+c)x + (b+c)x^2$$

This gives the linear system:

$$1 = a + b$$

$$0 = a + c$$

$$0 = b + c$$

Using your favorite method of solving linear systems (row reduction?), $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = -\frac{1}{2}$. So:

$$[T(b_1)]_\beta = [1]_\beta = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

The same method gives:

$$[T(b_2)]_\beta = [2x]_\beta = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}; \qquad [T(b_3)]_\beta = [1+2x]_\beta = \begin{pmatrix} 3/2 \\ -1/2 \\ 1/2 \end{pmatrix}$$

So:

$$[T]_\beta = \begin{pmatrix} 1/2 & 1 & 3/2 \\ 1/2 & -1 & -1/2 \\ -1/2 & 1 & 1/2 \end{pmatrix}$$

We can find $[T^2]_\beta$ by matrix multiplication: $[T^2]_\beta = ([T]_\beta)^2$. Or we can do the same thing as before: $T^2(b_1) = (1+x)'' = 0$, $T^2(b_2) = (1+x^2)'' = 2$, $T^2(b_3) = (x+x^2)'' = 2$. The answer is:

$$[T^2]_\beta = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

3.

(a) Let $\vec{x} \in N(U)$.
Then $U(\vec{x}) = \vec{0}_Z$.
Since T is onto, there exists $\vec{y} \in V$ such that $T(\vec{y}) = \vec{x}$.
Then $UT(\vec{y}) = U(T(\vec{y})) = U(\vec{x}) = \vec{0}_Z$.
Since UT is one-to-one, $\vec{y} = \vec{0}_V$.
So $\vec{x} = T(\vec{0}_V) = \vec{0}_W$.
Therefore, $N(U) = \{\vec{0}_W\}$, so U is one-to-one.

(b) We need to show:

U is one-to-one. This is given.

U is onto:
Let $\vec{z} \in Z$.
Since UT is onto, there exists $\vec{x} \in V$ such that $UT(\vec{x}) = \vec{z}$.
Let $\vec{y} = T(\vec{x})$. Then $U(\vec{y}) = U(T(\vec{x})) = UT(\vec{x}) = \vec{z}$.
So $\vec{z} \in R(T)$.
Therefore, $R(T) = Z$, so U is onto.

T is one-to-one.
Let $\vec{x} \in N(T)$.
Then $T(\vec{x}) = \vec{0}_W$.
Then $UT(\vec{x}) = U(T(\vec{x})) = U(\vec{0}_W) = \vec{0}_Z$.
Since UT is one-to-one, $\vec{x} = \vec{0}_V$.
Therefore, $N(T) = \{\vec{0}_V\}$, so T is one-to-one.

T is onto.
Let $\vec{y} \in W$.
Let $\vec{z} = U(\vec{y})$.
Since UT is onto, there exists $\vec{x} \in V$ such that $UT(\vec{x}) = \vec{z}$.
Then $U(\vec{y}) = U(T(\vec{x})) = \vec{z}$.
Since U is one-to-one, $\vec{y} = T(\vec{x})$, so $\vec{y} \in R(T)$.
Therefore, $R(T) = W$, so T is onto.

4. V and W are isomorphic if there exists an isomorphism $T : V \to W$. If V and W are finite-dimensional, V and W are isomorphic if and only if $\dim(V) = \dim(W)$.