Discussion Week 4: 2/16
MATH 110
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1. Let V be a three-dimensional vector space. For each subset of V below, decide which of the following terms you can conclude with certainty with the information given: Linearly Independent (LI), Linearly Dependent (LD), Generates V (G), Doesn't Generate V (DG), is a Basis for V (B).

   -S is linearly independent and has three elements.

   -T has four elements.

   -X is linearly dependent and has three elements.

   -Y has two elements.

   -Z generates V and has three elements.

2. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1 + a_2, a_1 - a_2)$. Prove that T is linear, and describe $N(T)$ and $R(T)$.

3. Define $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ by $T(A) = A + A^t$. Prove that T is linear. Describe $N(T)$ and $R(T)$.

4. Let $W = \{p \in P_n(\mathbb{R}) \mid \int_0^1 p(x)\,dx = 0\}$. Prove W is a subspace of $P_n(\mathbb{R})$, and find the dimension of W using rank-nullity. *Hint:* First prove that the function $T : W \to \mathbb{R}$ defined by $T(p(x)) = \int_0^1 p(x)\,dx$ is linear.

5. Without looking in the book, prove **Theorem 2.4:** Let V and W be vector spaces, and let $T : V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$.

6. **Challenge:** Let V, W be vector spaces, and $T : V \to W$ be linear. Suppose U is a subspace of W. Define $T^{-1}(U) = \{v \in V \mid T(v) \in U\}$. Prove $T^{-1}(U)$ is a subspace of V.

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Solutions:

1. Note: Any set with more than 3 elements is (LD). Any set with fewer than 3 elements is (DG). Any set with exactly three elements is either (LI), (G), (B), or it is (LD), (DG).

-(LI), (G), (B)

-(LD)

-(LD), (DG)

-(DG)

-(LI), (G), (B)

2.

Let $\vec{x} = (a_1, a_2)$, $\vec{y} = (b_1, b_2)$, $c \in \mathbb{R}$.

$$T(\vec{x} + c\vec{y}) = T((a_1, a_2) + c(b_1, b_2)) = T(a_1 + cb_1, a_2 + cb_2) =$$

$$= ((a_1 + cb_1) + (a_2 + cb_2), (a_1 + cb_1) - (a_2 + cb_2)) = ((a_1 + a_2) + c(b_1 + b_2), (a_1 - a_2) + c(b_1 - b_2)) =$$

$$= (a_1 + a_2, a_1 - a_2) + c(b_1 + b_2, b_1 - b_2) = T(\vec{x}) + cT(\vec{y})$$

So T is linear.

$(a_1, a_2) \in N(T)$ if and only if $T(a_1, a_2) = (a_1 + a_2, a_1 - a_2) = (0, 0)$. So $a_1 + a_2 = 0$ and $a_1 - a_2 = 0$ which implies $a_1 = a_2 = 0$. So $N(T) = \{\vec{0}\}$.

$(x_1, x_2) \in R(T)$ if and only if there exists $(a_1, a_2) \in \mathbb{R}^2$ such that $T(a_1, a_2) = (a_1 + a_2, a_1 - a_2) = (x_1, x_2)$. This has a solution with $a_1 = \frac{x_1 + x_2}{2}$ and $a_2 = \frac{x_1 - x_2}{2}$, so $R(T) = \mathbb{R}^2$.

3.

$$T(A + cB) = (A + cB)^t = A^t + (cB)^t = A^t. + cB^t = T(A) + cT(B)$$

So T is linear.

$A \in N(T)$ if and only if $T(A) = A + A^t = 0$, if and only if $A = -A^t$, if and only if A is antisymmetric.

$A \in R(T)$ if and only if A is symmetric- prove this as a challenge!

4. The proof that T is linear is similar to previous parts. Observe that, by definition, $W = N(T)$, so W is a subspace of $P_n(\mathbb{R})$. $R(T)$ is a subspace of $\mathbb{R}$, so $\dim(R(T)) = 0$ or 1. Since T is not the zero transformation, $\dim(R(T)) \neq 0$, so $\dim(R(T)) = 1$. Then:

$$\dim(N(T)) = \dim(P_n(\mathbb{R})) - \dim(R(T)) = (n+1) - 1 = n$$

5. See the book.