Discussion Week 3 Partial Solutions: 2/9
MATH 110
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1. Is the set $\{x^3 + x, x^2 + 1, 2x^3 - x^2 + 2x\}$ a linearly independent or linearly dependent subset of $P_3(\mathbb{R})$? Prove your answer.

2. If $\{u, v, w\}$ is linearly independent, show $\{u - v, v - w, u + w\}$ is linearly independent.

3. Prove Theorem 1.6 and its corollary (section 1.5, page 39)

4. Find a basis for the vector space $W = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$. What is its dimension?

5. **Challenge:** Find a basis for the set of $m \times n$ matrices such that the sum of the entries in any column equals the sum of the entries in any other column. What is its dimension?

Solutions:

1.

Suppose we have $a(x^3 + x) + b(x^2 + 1) + c(2x^3 - x^2 + 2x) = 0$. This simplifies to $(a + 2c)x^3 + (b - c)x^2 + (a + 2c)x + b = 0$. So we have:

$$a + 2c = 0$$

$$b - c = 0$$

$$a + 2c = 0$$

$$b = 0$$

Solving this linear system gives $a = b = c = 0$, so the set is linearly independent.

2.

Suppose we have $a(u-v)+b(v-w)+c(u+w) = 0$. This simplifies to $(a+c)u+(-a+b)v+(-b+c)w = 0$. Since $\{u, v, w\}$ is linearly independent, we have:

$$a + c = 0$$

$$-a + b = 0$$

$$-b + c = 0$$

Solving this linear system gives $a = b = c = 0$, so the set is linearly independent.

3.

If S_1 is linearly dependent, there exist distinct vectors $v_1, \ldots, v_n \in S_1$ and scalars $a_1, \ldots, a_n \in \mathbb{R}$, not all zero, such that $a_1 v_1 + \cdots + a_n v_n = 0$. Since $S_1 \subseteq S_2$, $v_1, \ldots, v_n \in S_2$. So S_2 is also linearly dependent.

For the converse, assume S_2 is linearly independent. If S_1 were linearly dependent, this would contradict Theorem 1.6, so S_1 is linearly independent.

4.

If $x \in W$, $x = (x_1, \ldots, x_n)$ with $x_1 + \cdots + x_n = 0$. Then $x_1 = -x_2 - x_3 - \cdots - x_n$. So (using column vector notation):

$$x = \begin{pmatrix} -x_2 - x_3 \cdots - x_n \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Challenge.

The dimension is $mn - (n-1)$. As a hint, introduce the variable S to denote the common sum of all the columns.