Discussion Week 2 Partial Solutions: 2/2
MATH 110
GSI: Alex Zorn

1. Each of the following claims is **false**. Prove this.

1. $\text{span}(S_1) \cap \text{span}(S_2) = \text{span}(S_1 \cap S_2)$ for any subsets S_1, S_2 of a vector space V.

2. $\text{span}(S_1) \cup \text{span}(S_2) = \text{span}(S_1 \cup S_2)$ for any subsets S_1, S_2 of a vector space V.

3. Any subset of $\mathbb{R}^2$ that contains the zero vector and is closed under scalar multiplication is a subspace of $\mathbb{R}^2$.

4. Any subset of $\mathbb{R}^2$ that contains the zero vector and is closed under addition is a subspace of $\mathbb{R}^2$.

2. Prove that the span of $\{(1, 1, 0), (1, -1, 0)\}$ is the set $W = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$.

Challenge. Theorem 1.5 says that if S is a subset of V, then $\text{span}(S)$ is a subspace of V, $S \subseteq \text{span}(S)$, and if W is any subspace of V containing S, then $\text{span}(S) \subseteq W$. Use this theorem to prove the following *without writing any linear combinations.*

1. A subset W of a vector space V is a subspace of V if and only if $\text{span}(W) = W$.

2. If $S_1 \subseteq S_2$ then $\text{span}(S_1) \subseteq \text{span}(S_2)$.

3. For any two subsets S_1, S_2, $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.

—

Solutions:

1.1. Let $V = \mathbb{R}^2$, $S_1 = \{(1, 0)\}$, $S_2 = \{(2, 0)\}$. Then $\text{span}(S_1) = \text{span}(S_2) = \{(x, 0) \mid x \in \mathbb{R}\}$, so $\text{span}(S_1) \cap \text{span}(S_2) = \{(x, 0) \mid x \in \mathbb{R}\}$. But $S_1 \cap S_2 = \emptyset$, so $\text{span}(S_1 \cap S_2) = \{(0, 0)\}$.

1.2. Let $S_1 = \{(1, 0)\}$, $S_2 = \{(0, 1)\}$. Then $\text{span}(S_1 \cup S_2) = \mathbb{R}^2$, but $\text{span}(S_1) \cup \text{span}(S_2)$ is just the union of the x and y axes.

1.3. Take, for example, $\text{span}(S_1) \cup \text{span}(S_2)$ where $S_1 = \{(1, 0)\}$ and $S_2 = \{(0, 1)\}$.

1.4. Consider $W = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}$.

2. Let $V = \text{span}(\{(1,1,0),(1,-1,0)\})$.

To prove $V \subseteq W$:
Let $\vec{x} \in V$. Then $\vec{x} = a(1,1,0) + b(1,-1,0) = (a+b, a-b, 0)$, so $\vec{x} \in W$.

To prove $W \subseteq V$:
Let $\vec{x} \in W$. Then $\vec{x} = (x, y, 0)$ for some $x, y \in \mathbb{R}$.
We want to find a, b such that $\vec{x} = a(1,1,0) + b(1,-1,0) \Leftrightarrow (x,y,0) = (a+b, a-b, 0)$. This has a solution: $a = \frac{x+y}{2}$, $b = \frac{x-y}{2}$. So $\vec{x} \in V$.

Challenge 1. First assume W is a subspace of V. $W \subsetneq \text{span}(W)$ for any set W. Since W is a subspace and $W \subseteq W$, $\text{span}(W) \subseteq W$. So $W = \text{span}(W)$.

Now assume $\text{span}(W) = W$. Since W is a subspace, $\text{span}(W)$ is a subspace.