Discussion Week 1 Partial Solutions: 1/26
MATH 110
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Prove that each of the following is or is not a vector space over $\mathbb{R}$:

**1.** $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$.

**2.** $U$ is the set of all real-valued functions on the real line satisfying $f(t) < 1$ for all $t$.

**3.** $V$ is the set of all $2 \times 2$ matrices whose trace equals zero.

**4.** $X = \{(x, y, z) \in \mathbb{R}^3 \mid \text{at least one of } x, y, z = 0\}$.

**5.** $T = \{p \in P_3(\mathbb{R}) \mid p(1) = 0\}$.

**Challenge 1.** Let $v$ be a nonzero element of a vector space $V$. Show that the set $\{cv \mid c \in \mathbb{R}\}$ is a subspace of $V$. These subspaces are called **one-dimensional**.

**Challenge 2.** If $W$ is a subspace of $\mathbb{R}^2$, show that either $W = \{0\}$, $W = \mathbb{R}^2$, or $W$ is one-dimensional.

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**Solutions**

**1.**

This is a vector space, as it is a subspace of $\mathbb{R}^3$.

Proof: $W$ contains the zero vector. Assume $\vec{x} \in W$ and $\vec{y} \in W$, and $t \in \mathbb{R}$. Write $\vec{x} = (a_1, b_1, c_1)$, $\vec{y} = (a_2, b_2, c_2)$. We know $a_1 + b_1 + c_1 = 0$ and $a_2 + b_2 + c_2 = 0$.

Then $\vec{x} + t\vec{y} = (a_1 + ta_2, b_1 + tb_2, c_1 + tc_2)$, and:

$$(a_1 + ta_2) + (b_1 + tb_2) + (c_1 + tc_2) = (a_1 + b_1 + c_1) + t(a_2 + b_2 + c_2) = 0 + t \cdot 0 = 0$$

So $\vec{x} + t\vec{y} \in W$.

**2.**

This is not a vector space. For example, the constant function $f(t) = 1/2$ is in $U$, but $3f$ is not in $U$, so $U$ is not closed under scalar multiplication.

**3.**

This is a vector space, proof similar to 1.

**4.**

This is not a vector space. For example, $(1, 0, 0) \in X$ and $(0, 1, 1) \in X$, but $(1, 0, 0) + (0, 1, 1) = (1, 1, 1) \notin X$, so $X$ is not closed under addition.

**5.**

This is a vector space, proof similar to 1.