Midterm 1 Concepts Review
MATH 110
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Disclaimer: This is not intended to be comprehensive- it is just a list of a few of the things I think are important. If something is not on this review sheet it does **NOT** mean that it won't be on the midterm.

I. Terminology

Know the following definitions:

Vector Space; Subspace

General Set Terminology: Union, Intersection, Subset, Empty Set

Sums: If W_1 and W_2 are subspaces of V, what is W_1+W_2. What does it mean to say $V = W_1 \oplus W_2$?

Sets of Vectors: Let S be a set of vectors in a vector space V. Know the meaning of: span(S), S is linearly independent, S generates V, S is a basis of V.

Dimension: Finite-dimensional, Infinite-dimensional, Dimension.

General Function Terminology: Domain, Codomain, One-to-one, onto, composition.

Linear Transformations: What is a linear transformation? If T is a linear transformation: $N(T)$, $R(T)$, the kernel of T, the null space of T, the range of T, the image of T, the nullity of T, the rank of T. (Some of these are the same thing).

Isomorphisms: Invertible linear transformation, Invertible matrix, isomorphism, isomorphic vector spaces.

Coordinates/Matrices: Trace of a matrix, transpose of a matrix, Coordinate vector, Matrix representation of a linear transformation, change-of-coordinates matrix, the linear transformation L_A when A is a matrix.

II. Examples

Understand the following examples of vector spaces well. When they are finite-dimensional, know their dimension and an example of a basis.

Vector Spaces: $\mathbb{R}^n$, $P_n(\mathbb{R})$, $P(\mathbb{R})$, $\mathcal{F}(S,\mathbb{R})$, $M_{m\times n}(\mathbb{R})$, $\mathcal{L}(V,W)$, V^*.

III. Important Theorems

Most of the important non-trivial theorems so far have to do with bases and dimension:

Theorem 1: Let V be finite-dimensional, say $\dim(V) = n$. Then:

- Any basis has exactly n vectors.
- Any linearly independent set has $\leq n$ vectors and can be extended to a basis for V, so any linearly independent set with exactly n vectors is a basis.
- Any set that generates V has $\geq n$ vectors and can be reduced to a basis for V, so any set that generates V and has exactly n vectors is a basis.

Theorem 2: Let V be finite-dimensional, and $W \subseteq V$. Then W is finite-dimensional, $\dim(W) \leq \dim(V)$, and if $\dim(W) = \dim(V)$ then $W = V$.

Theorem 3: Suppose W_1, W_2 are finite-dimensional subspaces of a vector space V. Then $W_1 + W_2$ is finite-dimensional, and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

Theorem 4: Suppose $T : V \to W$ is a linear transformation and V and W are finite-dimensional. Then:

- T is one-to-one iff $N(T) = \{0\}$ iff $\text{nullity}(T) = 0$
- T is onto iff $R(T) = W$ iff $\text{rank}(T) = \dim(W)$
- $\text{nullity}(T) + \text{rank}(T) = \dim(V)$

Theorem 5: If V and W are finite-dimensional vector spaces then $\dim(V) = \dim(W)$ if and only if V and W are isomorphic. If $\dim(V) = \dim(W)$ and $T : V \to W$ is a linear transformation, the following are equivalent:

- T is one-to-one
- T is onto
- T is invertible

Theorem 6: Let V be a finite-dimensional vector space, $\beta = \{v_1, \dots, v_n\}$ a basis for V, and W any vector space. If $w_1, w_2, \dots, w_n$ is any list of vectors in W (not necessarily linearly independent, a basis, etc), then there exists a unique linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for $1 \leq i \leq n$.

We also have some important theorems about coordinate vectors and matrix representations. Suppose V, W, Z are finite-dimensional vector spaces with dimensions m, n, p respectively, and ordered bases α, β, γ respectively.

Theorem 7: If $\vec{x} \in V$ and $T : V \to W$ is linear, $[T]_\alpha^\beta [\vec{x}]_\alpha = [T(\vec{x})]_\beta$. If $U : W \to Z$ is linear, $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$.

Theorem 8: The function $\phi : V \to \mathbb{R}^n$ defined by $\phi(x) = [x]_\beta$ is an isomorphism. The function $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(\mathbb{R})$ defined by $\Phi(T) = [T]_\beta^\gamma$ is an isomorphism.

Theorem 9: Let α be the standard ordered basis for $\mathbb{R}^n$ and β be the standard ordered basis for $\mathbb{R}^m$. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, and $A = [T]_\alpha^\beta$, then $T = L_A$. Conversely, if A is an $m \times n$ matrix, and $T = L_A$, then $A = [T]_\alpha^\beta$.

IV. Computations/Simple Proofs You Should Be Able To Do.

Most of these have been tested on the homework (relevant sections listed):

Prove that a subset of a vector space is/is not a subspace (1.3)

Prove that a subset S of a vector space is/is not linearly independent, or that it generates V (or perhaps a subspace W of V). (1.4, 1.5).

Finding a basis for a subspace, and proving it's a basis. Finding the dimension of a subspace. (1.6).

Proving that a function is/is not a linear transformation, that it is/is not one-to-one or onto or an isomorphism, finding a basis for the null space/range. (2.1, 2.4)

Finding the coordinate vector of a vector w/r/t a basis, finding the matrix representation of a transformation w/r/t a basis of the domain and a basis of the codomain, finding the change-of-coordinates matrix between two bases (2.2, 2.5).

Multiplying two matrices (2.3).

Finding a dual basis (2.6).