Thesis errata

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The principal purpose of this note is to correct the following key error in Chapter 3 of my thesis [Zef25]: if $E/\mathbb{Q}_p$ is a finite extension, the structure map $p : \underline{E} \to *$ is not cohomologically smooth. Indeed this is quite obvious: with conventions as there, so writing $\mathbf{1}$ for the unit object $\mathbb{Q}_\ell$, we have $p^*\mathbf{1} = \mathbf{1}$ the sheaf $C_c(\underline{E}, \mathbb{Q}_\ell)$ of compactly supported functions on $\underline{E}$, while $p^!\mathbf{1} \simeq \mathrm{Dist}(\underline{E}, \mathbb{Q}_\ell)$, smooth linear functionals on the space of compactly supported functions, following [HKW22, Example 4.2.1].

However, since $\underline{E}$ is not only a locally profinite set but a unimodular locally profinite group, it carries a unique Haar measure up to scaling; for concreteness, we can even normalize the Haar measure such that $\underline{\mathcal{O}_E}$ has volume 1, giving a fixed choice of Haar measure. We can then hope to obtain a map from compactly supported functions to distributions by integrating against these functions. More abstractly, we can view integrating against our fixed Haar measure as a map $\int : p_!p^*\mathbf{1} \to \mathbf{1}$, inducing a map $p^*\mathbf{1} \to p^!\mathbf{1}$. As in [FK24, §3.4], this induces a natural transformation $p^* \to p^!$; this can also be seen directly as above.

In particular, although p is not cohomologically smooth, it is cohomologically quasi-smooth of relative dimension 0. This allows us to replace statements about the smoothness of Banach–Colmez spaces with quasi-smoothness, which (often after some additional work) will be enough.

We note some of the places that cohomological smoothness is assumed, which need to be modified. In general, we neglect issues specific to the setting of $\mathcal{O}_F$-module stacks discussed in §3.3.2 (and the corresponding implementation in §3.5), and treat the corresponding issues only after passing to infinite level $K \to \{1\}$; a more careful inspection should reveal analogous issues and fixes.

- In Proposition 3.3.1, all stacks are assumed to be cohomologically smooth over the base. This is used to allow the same assumptions in Propositions 3.3.2 and 3.3.3.
  - In Proposition 3.3.2, smoothness is used to avoid assumptions of global presentability as in [FYZ23]. In particular it is shown that maps of smooth stacks in E-vector spaces factor as the composition of a smooth map and a closed immersion. However we can actually show that a map of Banach–Colmez spaces factors into closed immersions and smooth maps (even surjections) without this assumption, using the inclusion $\underline{E} = \mathcal{BC}(\mathcal{O}) \to \mathcal{BC}(\mathcal{O}(1))$ to a smooth space induced by a fixed nonzero section of $\mathcal{O}(1)$, which can be viewed as corresponding to a fixed characteristic zero algebraically closed complete extension C of $\mathbb{Q}_p$. (One uses this to prove that the zero section can always be factored in this way, thence the graph morphism, and finally the projection via the construction of Banach–Colmez spaces as quotients by $\underline{E}$-vector spaces of extensions of disks by E-vector spaces.)
  - In Proposition 3.3.3, in the statement the assumption of smoothness on the stacks is only necessary to deduce the quasi-smoothness of f; for the proof, the factorization as above suffices.

- Some of the proofs of the geometric properties of Proposition 3.5.7 use the smoothness of Banach–Colmez spaces, which in general needs to be weakened to quasi-smoothness. These are:
  - Part (d): it can instead be checked by hand that (after passing to $K = \{1\}$) the diagrams are cohomologically pullable, i.e. the morphism from the top-left corner to the pullback is cohomologically quasi-smooth.
  - Part (j): one can instead dualize the squares and check that the dual is pushable. This requires some care for the analogue of [FYZ23, Lemma 7.2.1] in our setting, as the proof does not directly translate, but an analogous idea can be made to work for the statement we need.
  - Part (m): similar to (d).

To apply [FK24, Theorems 5.4.2 and 5.4.3], we need smoothness of $\widetilde{U} \to \mathrm{Bun}_H$ and $\widetilde{U}^\perp \to \mathrm{Bun}_H$ and properness of $j^U \to */K$. The first can be checked directly with some restrictions on the space of shtukas (see below), but the proof given of the second in Proposition 3.5.7 (c) is insufficient. The issue can be avoided by amending the definition of the special cycle $[\mathcal{Z}^r_{\widetilde{\mathcal{E}},\widetilde{\mathfrak{E}},\eta,C}]$ to be the pairing of the pushforwards of the pullbacks of the special correspondences $\mathfrak{s}$ and μ, rather than first pairing and then pulling and pushing. This is slightly less satisfactory from the point of view of the analogy, and one might guess that the results should be the same either way; but this lets us avoid the use of these theorems entirely, which also lets us safely pass to the case $K = \{1\}$ and so neglect issues of $\mathcal{O}_F$-module stacks.

To form the pairing (i.e. take the trace at the end) we need the pushforward of the constant sheaf along $\widetilde{U} \to \mathrm{Bun}_H$ to be suave over the base. This holds so long as $\widetilde{U} \to \mathrm{Bun}_H$ is smooth, so we return to this question: this does not hold in general, since $\mathcal{F}_{\mathrm{univ}}$ on Bun_H may have slope zero summands, in which case the corresponding summand of $\widetilde{U}$ is non-smooth.

To resolve this issue, we restrict to the locus in Bun_H consisting of H-bundles with no summands of slope zero, over which $\widetilde{U}$ is smooth. We can recover the shtuka stack by taking it to be the base change of $\mathrm{Hk}^r_H \to \mathrm{Bun}_H \times \mathrm{Bun}_H$ along $* \to \mathrm{Bun}_H \times \mathrm{Bun}_H$ given by the pair of H-bundles $(\mathcal{E}^b, \mathcal{E}^{b'})$ where $b, b' \in B(H)$ are chosen such that $\mathcal{E}^b$ and $\mathcal{E}^{b'}$ have no slope zero summands, e.g. b, b' basic and nontrivial. Note that this does exclude the case $b = 1$, so the definition used in [Zef25] needs some modification; but this is straightforward.

References

[FK24] Tony Feng and Adeel Khan. Modularity of higher theta series II: Chow group of the generic fiber. *arXiv preprint arXiv:2403.19711*, 2024.

[FYZ23] Tony Feng, Zhiwei Yun, and Wei Zhang. Modularity of higher theta series I: cohomology of the generic fiber. *arXiv preprint arXiv:2308.10979*, 2023.

[HKW22] David Hansen, Tasho Kaletha, and Jared Weinstein. On the Kottwitz conjecture for local shtuka spaces. In *Forum of Mathematics, Pi*, volume 10, page e13. Cambridge University Press, 2022.

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