Practice midterm 4

Complex analysis, lecture 4

December 1, 2025

As usual, be sure to include your method, and remember to write your name.

Problem 1. Using the argument principle around a circle of sufficiently large radius R, show that $p(z) = z^5 - z + 3$ has five zeros (counting multiplicity), as predicted by the fundamental theorem of algebra.

The above problem is directed towards Objective 11 (the argument principle).

Problem 2. Use Rouché's theorem to show that $p(z) = z^5 - z + 3$ has *no* zeros with $|z| < 1$.

The above problem is directed towards Objective 11 (the argument principle).

Problem 3. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function, where $\mathbb{D} = \{z : |z| < 1\}$ is the unit disk. Suppose that $f(1/2) = f'(1/2) = -1/2$. Show that f cannot be a conformal map.

The above problem is directed towards Objective 12 (the Schwarz lemma).

Problem 4. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function whose restriction to $\mathbb{D} = \{z : |z| < 1\}$ has image in $\mathbb{D}$ and gives a conformal map $\mathbb{D} \to \mathbb{D}$. Show that f must be of the form $f(z) = e^{i\theta} z$ for some real number θ.

The above problem is directed towards Objective 12 (the Schwarz lemma).

Problem 5. Solve the Dirichlet problem on the disk $D = \{z : |z| < 2\}$ of radius 2 centered at the origin: if $h : \partial D \to \mathbb{C}$ is a continuous function, give an integral formula for the unique harmonic function $\tilde{h} : D \to \mathbb{C}$ with boundary values h.

The above problem is directed towards Objective 13 (the Poisson integral formula and harmonic functions).

Problem 6. Let $h(z) = e^{|z|^2}$, as a function $\mathbb{C} \to \mathbb{C}$. Show that h does not satisfy the mean value property.

The above problem is directed towards Objective 13 (the Poisson integral formula and harmonic functions).

Problem 7. Let $\Omega_1 = \{z \neq 0 : \frac{\pi}{3} < \arg z < \pi\}$ and $\Omega_2 = \{z : |z| < \pi\}$. Find a conformal map $\Omega_1 \to \Omega_2$.

The above problem is directed towards Objective 14 (the Riemann mapping theorem).

Problem 8. Does there exist a simple connected domain $\Omega \subseteq \mathbb{C}$ with a conformal map $\Omega \to \mathbb{C}$ and a conformal map $\Omega \to \mathbb{D}$, where $\mathbb{D} = \{z : |z| < 1\}$ is the unit disk? Give an example if so, or prove that it is impossible if not.

The above problem is directed towards Objective 14 (the Riemann mapping theorem).