Project guidelines

Complex analysis, lecture 4

1. Timeline and requirements

The goal of the project is to (1) investigate some problem using the mathematical concepts we've studied in this class and (2) write an expository paper on the topic, i.e. explain it in detail to an audience unfamiliar with it. Time permitting, you may also optionally (3) give a short presentation to the class on your findings.

This is a fairly open-ended project; you may use any resources you like, and the converse of this is that it is your job to find (and properly cite) references to understand your desired topic. That said, I am happy to help you find resources if you are having difficulty, especially on more obscure topics. (You don't have to notify me about your proposed topic, but if I don't hear from you I'll assume you are on top of finding sources etc.)

The project will be due by the end of the day **December 5, 2025**. If you would like to do a presentation, please let me know by **November 24**, including your planned topic (you can change this later).

Guidelines

The primary goal of this project is to understand the mathematics of your topic; nearly as important however is clearly communicating that understanding. Imagine that you are trying to explain the concepts you have studied to someone who has a similar amount of background to you, but has not necessarily studied these particular topics.

Like any paper, in addition to the main body of the exposition your paper should include a short introduction, explaining the main ideas, motivation, and background of your paper, as well as a list of sources. (The specific formatting of your sources does not matter so long as it is clear.)

All papers should be typed.[1] I encourage you to use LaTeX[2], but it is not required, and you may use whatever software you prefer.

There is no hard guideline for the length of your papers: they should be the length they need to be in order to concisely and clearly explain your topic in detail to the appropriate audience, but in practice I am expecting probably 1-2 pages on the shorter end or 4-5 on the longer end. Precise details of formatting are up to you.

I am open to the idea of projects which do not take the form of a paper, or not solely that form; for example, they could have a coding or graphic component. If you want to do something like this though please discuss it with me first.

[1] If this is a particular hardship for you, we can discuss alternatives.

[2] LaTeX is a software system for creating documents, especially those involving large numbers of mathematical or scientific symbols, and is probably what virtually all mathematical documents you have encountered at least in college were written in, including this one; there are many editors available, including online ones such as `overleaf.com`.

Finally, note that as usual in this class the use of generative AI in writing your project is not permitted. However, you may find it useful as a research tool, or to help you understand a topic; if you do use it, make sure to check its sources and statements, as it is particularly good at convincingly justifying claims whether or not they are true (not a skill I want you to pick up!).

Grading

A successful project will be worth some number of challenge points, most likely between 1 and 5, possibly more in exceptional cases. Projects will be graded for mathematical correctness and depth/scope.[3] For example, a completely correct paper which solves a very small problem might only be worth one or two points; this should be viewed as a fully successful paper! To earn more points, a project could aim for a larger scope, but note that an ambitious but incorrect paper is less successful than a modest but correct one, and will earn fewer points if any. My expectation is that a project of moderate scope without significant errors should be worth 3-4 points. Some topics are suggested below.

If you choose a topic which is particularly closely related to one of our objectives, you may choose to have the project counted towards your homework grade for that objective. In this case, you should specify this with your submission, as well as which objective you would like it counted for; the project will then be graded on the S/P/N scale, and that mark will replace your homework grade for that objective (if better than the current mark). If I don't think that the project is close enough to that objective, after discussing with you I reserve the right to instead grade it for challenge points as above.

2. Topic suggestions

Any of these should be taken as a collection of related possible ideas around which to base your project; you do not necessarily need to cover everything mentioned, and might cover aspects not mentioned. Some are closely related to some of our objectives or otherwise covered in Gamelin, others are not.

Fluid dynamics. Look into the basics of fluid dynamics and applications of complex-analytic methods to problems therein, as discussed for example in sections III.6 and XI.4 in Gamelin. See what other applications you can find elsewhere, and work out some examples.

The prime number theorem. The prime number theorem gives an estimate for the number of primes less than or equal to a positive real number x (it is approximately $\frac{x}{\log x}$, for x large). This is proven by relating prime numbers to certain complex-analytic functions, and then by integrating along carefully chosen contours one can recover quantities such as the count of prime numbers (or related variants) from the residue formula. Look into this; although the full proof of the prime number theorem may be too big a topic, sketch some of

[3] I encourage you to write clearly and concisely, but these will not be assessed for grading.

the ideas, and in particular explain where the complex-analytic ideas come in and how they are used.

One version of this story is given in section XIV of Gamelin. Other versions can be found elsewhere. In particular, Gamelin uses a "Tauberian theorem," which via a trick due to Newman and Zagier makes the proof significantly faster, but arguably obscures some of the complex analysis underlying the argument; you may also want to look at the more traditional argument of Hadamard and de la Vallée Poissin, which gives an explicit formula for a certain prime-counting function using the residue theorem.

Hyperbolic geometry. In our discussion of the Schwarz lemma and Pick's lemma, we mentioned some applications to hyperbolic geometry. Look into this further, as for example in Gamelin section IX.3, and more generally explore other features of hyperbolic geometry, its applications and connections to other fields, generalizations, other models, etc.

The Dirichlet problem. We have studied the problem of finding harmonic functions with certain properties, e.g. harmonic conjugates, and in objective 13 will study the problem of finding harmonic functions on the disk with given values on its boundary. This is the Dirichlet problem on the disk. More generally, on any domain D we can study the problem of how to extend a given function on ∂D to a harmonic function on D; this is the Dirichlet problem on D. Research its solution and applications; one version of this story is in section XV of Gamelin, which in particular explains how the Dirichlet problem can be applied to give a proof of the Riemann mapping theorem. There are also many applications and generalizations which you may wish to discuss.

Riemann surfaces. We have discussed some Riemann surfaces in this class, especially in objective 2, but they are a substantially deeper topic than we have had time to cover. Section XVI of Gamelin gives a more general perspective and proves a uniformization theorem. More generally there is a great deal to be said about Riemann surfaces; say some of it.

The Fourier transform. For any bounded domain D with piecewise smooth boundary and a smooth function $f : D \cup \partial D \to \mathbb{C}$, we can define its "Cauchy integral transform"

$$z \mapsto \int_{\partial D} \frac{f(w)}{w - z} \, dw.$$

Cauchy's integral formula can then be rephrased as the theorem that if f is analytic on D, the Cauchy integral transform of f, viewed as a function on D, is just f again.

More generally, we could define many other integral transforms. Perhaps the most important of these is the Fourier transform: if f is defined and continuous on an interval $[a, b]$, we can define its Fourier transform on $\mathbb{C}$ given by

$$z \mapsto \int_a^b f(t) e^{-itz} \, dt.$$

This is defined in Gamelin Exercise IV.6.2. It can also be defined much more generally. Look into some of its definitions, properties, and applications, of which there are many.

Also relevant is the Fourier *series*, introduced in Gamelin section VI.6. Look into some of its properties and applications, and discuss how it is related to the Fourier transform.

Several complex variables. The subject we call "complex analysis" could also be referred to as the study of complex functions of a single variable. This phrasing suggests the study of complex functions of more than one variable. Many things can be defined analogously, but some of the behavior is qualitatively different. Describe some of the basic ideas and problems in the study of several complex variables, and discuss the solutions to some of these problems in the one-variable case in terms of results we've seen in this class.

Complex geometry. This can be viewed as a generalization of the previous topic, but in practice is often treated distinctly. For those with some experience with manifolds: recall that a manifold can be described as a topological space which is locally isomorphic to $\mathbb{R}^n$ for some n. A complex manifold should analogously be locally isomorphic to $\mathbb{C}^n$ for some n. But note that differentiable functions on $\mathbb{R}$ behave, as we have seen, very differently from differentiable functions on $\mathbb{C}$; so requiring our transition functions to be holomorphic makes complex manifolds a different sort of beast from real manifolds; their study is the study of complex geometry. Discuss some of this, and some of the basic ideas and problems in complex geometry, keeping in mind $\mathbb{C}$ as the simplest example of a complex manifold.

Choose your own. Find your own topic! It should be related to the material from this class, so using the methods of complex analysis, holomorphic functions, etc. Otherwise you are free to choose any topic that interests you, using the above as a guide. If you choose your own topic, I suggest discussing it with me so I can warn you if e.g. it seems too hard or not close enough to our class, but this is not required.

3. Tips

Based on looking through some past comments I've given student projects, here are some things to look out for:

- Make sure to define your terms. If defining all the terminology you're using is especially laborious, consider whether you really need this terminology—do you really use it? If not, how could you cleanly cut it out?
- For technical definitions and arguments that you do include, make sure that they serve the overall thrust of the paper. On the other hand, make sure not to omit so many technical details that the paper loses its technical content!
- Make clear which statements you are proving, which are known but are not proven in your paper, and which may be unknown.
- In terms of writing, don't overuse symbols: it is generally better to write things out rather than abbreviate. Use symbols only when they are the clearest way to write what you need to, e.g. for formulas.
- Your paper should not read like a collection of bullet points: each paragraph should be coherent. More generally, your paper should not be a collection of facts, but flow

naturally, including motivation, results, proofs, and possibly discussions.

- In terms of citations, although the particular citation format doesn't matter, it should always be clear what the sources of your ideas and material are. For example if you include graphics which you did not generate, these should be cited. If you are mostly using one source, you can indicate this at the beginning and then put citations where other sources are used, there's no need to be constantly citing the same source every sentence. When you do cite sources, especially long ones, make your citations as precise as possible, e.g. theorem or page number.