Lecture 9: harmonic functions, reprise

Complex analysis, lecture 4
September 24, 2025

Last time, we discussed path integrals and their independence of path. In particular, we saw that the integral of a differential along a path in a domain is independent of the path if (and only if) the differential is exact, and exact differentials are closed, giving a condition we can check easily. In general, closed differentials are not necessarily exact, but on star-shaped domains they are, so in this case closed and exact are equivalent.

Today, we'll apply this result to prove that harmonic functions on star-shaped domains really do have harmonic conjugates, a result we've previously referred to but didn't formally prove. We'll then discuss two more important properties of harmonic functions: the mean value property and the maximum principle.

1. Harmonic conjugates

Suppose $u : D \to \mathbb{R}$ is a harmonic function, where $D \subset \mathbb{R}^2$ is a domain. We claim that the differential

$$\omega = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$$

is closed. Indeed, recall that a differential $P\,dx + Q\,dy$ is closed if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, so here we are asking for

$$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right),$$

i.e. that

$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2},$$

which is exactly Laplace's equation $\Delta u = 0$.

By our results from last time, it follows that if D is a star-shaped domain, there exists some function $v : D \to \mathbb{R}$ such that $\omega = dv$; that is, since ω is closed on a star-shaped domain, it must be exact. Therefore $\omega = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy = dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy$, so

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.$$

These are exactly the Cauchy–Riemann equations for u and v, so $u + iv$ is analytic, i.e. v is a harmonic conjugate for u! Explicitly, since $dv = \omega$, we immediately get the formula

$$v(B) = \int_A^B d\omega = \int_A^B -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy,$$

which agrees with the formula we sketched when first talking about harmonic conjugates.

2. The mean value property

Suppose $f : D \to \mathbb{R}$ is a continuous function on a domain $D \subset \mathbb{R}^2$. If $P = (x_0, y_0) \in D$ and γ is the circle around P of radius r, i.e. $\gamma = \{Q \in D : |Q - P| = r\}$ for any r small enough that γ is contained in D, we can consider the "average value" of f on the circle:

$$A(r) = \frac{1}{2\pi} \int_0^{2\pi} f(x_0 + r\cos t, y_0 + r\sin t)\, dt.$$

(If we viewed D as a subset of $\mathbb{C}$ and $P = z_0 = x_0 + iy_0$, we could write this as $f(z_0 + re^{it})$.) This is a continuous function of r so long as r is not too high; D must contain some disk around P of radius R, so this is well-defined for $r < R$. As $r \to 0$, each point on which we're estimating f approaches P and so $A(r) \to f(P)$. In general though for $r > 0$ the average value $A(r)$ may differ from $f(P)$, e.g. if P is a local maximum.

If f is harmonic, however, we claim that the average value at any radius $r < R$ is *equal* to $f(P)$ on the nose. That is: if f is harmonic on D, then

$$f(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} f(x_0 + r\cos\theta, y_0 + r\sin\theta)\, d\theta.$$

This is quite a special property: it means we can detect the value on the center of the loop only by evaluating the function on the perimeter!

To see this, we again use the fact that $-\frac{\partial f}{\partial y}\, dx + \frac{\partial f}{\partial x}\, dy$ is closed if f is harmonic. It then follows from Green's theorem that

$$\int_\gamma -\frac{\partial f}{\partial y}\, dx + \frac{\partial f}{\partial x}\, dy = 0.$$

Parametrizing the circle by $\gamma(t) = (r\cos t, r\sin t)$ for $0 \le t \le 2\pi$, this integral is

$$\begin{aligned} r\int_0^{2\pi} \left(\frac{\partial f}{\partial x}\cos t + \frac{\partial f}{\partial y}\sin t\right) d\theta &= r\int_0^{2\pi} \frac{\partial f}{\partial r}(x_0 + r\cos t, y_0 + r\sin t)\, dt \\ &= r\frac{\partial}{\partial r}\int_0^{2\pi} f(x_0 + r\cos t, y_0 + r\sin t)\, dt \\ &= 2\pi r\frac{\partial}{\partial r}A(r) \\ &= 0, \end{aligned}$$

i.e. $A'(r) = 0$. Hence $A(r)$ is constant for $0 < r < R$, and since it is continuous at 0 we must have $A(r) = \lim_{r\to 0} A(r) = f(x_0, y_0)$ for all $r < R$.

More generally, we'll say that f has the mean value property at (x_0, y_0) if $A(r) = f(x_0, y_0)$ for any $r < R$, and that f has the mean value property on D if it has the mean value property everywhere in D. So we can restate the above as the fact that harmonic functions have the mean value property. We will, hopefully, eventually see that this is a characterizing property: any function f satisfying the mean value property is harmonic. Note that a priori, f is only continuous, while harmonic functions have to be twice differentiable, so this is actually quite surprising!

3. The maximum principle

Recall the fact that a continuous function f on a compact set S is bounded and attains its maximum. A domain D, however, is (at least in general) not compact, since it is open, while compact sets are closed. (One can have sets which are both open and closed, but not interesting ones in our setting.) If u is a function on D which is further harmonic, we claim that almost the opposite is true: u will never attain its maximum, unless it is constant. More precisely, if $u : D \to \mathbb{R}$ is a harmonic function and for some $M \in \mathbb{R}$ we have $u(z) \leq M$ for all $z \in D$, then if $u(z_0) = M$ for $z_0 \in M$, then u is constant. This is sometimes called the strict maximum principle (for real-valued harmonic functions).

The proof uses some topology, which I'll just briefly summarize: since D is a domain, i.e. an open and (path-)connected subset of the plane, it can't be partitioned into two nonempty open sets (indeed, this is the definition of being connected). So we'll write down two disjoint open sets inside D whose union is D, and then conclude that one of them has to be empty.

First, let $U \subseteq D$ be the set of points $z \in D$ such that $u(z) < M$. This is an open condition, so it's an open set. Its complement $V \subseteq D$ is the set of points $z \in D$ such that $u(z) \geq M$; since we know that $u(z) \leq M$, this is equivalently the set of points with $u(z) = M$. We'd like to show that this is open. If we knew this, then the above argument would tell us that either V is empty, so $U = D$ and u doesn't attain its maximum on D; or W is empty, so $u(z) = M$ for every $z \in D$, i.e. u is constant.

To show that V is open, we need to show that for every $z_0 \in V$, we can find an open disk of some radius $R > 0$ centered at z_0 which is contained in V. For $z_0 \in V$, we have $u(z_0) = M$. Since u is harmonic, it satisfies the mean value property, so there is some $R > 0$ such that for any $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} \left(u(z_0) - u(z_0 + (r\cos t, r\sin t))\right)\, dt.$$

Since $u(z) \leq M$ for all $z \in D$, the integrand $u(z_0) - u(z_0 + (r\cos t, r\sin t))$ is nonnegative for all t; so the only way the integral can be 0 is if in fact $u(z_0) - u(z_0 + (r\cos t, r\sin t)) = 0$ for all t. Hence for every z contained in the open disk of radius R centered at z_0, we can write it as $z_0 + (r\cos t, r\sin t)$ for some $r < R$ and $0 \leq t \leq 2\pi$, hence $u(z) = u(z_0) = M$, i.e. $z \in V$; so V contains this disk, and therefore is an open set, so we're done.

A consequence is the strict maximum principle for complex harmonic functions: if h is a bounded harmonic complex-valued function on D such that $|h(z)| \leq M$ for all $z \in D$ and $|h(z_0)| = M$ for some $z_0 \in D$, then h is constant on D. This is not quite immediate, since $|h(z)|$ isn't necessarily harmonic. However, the real and imaginary parts of h are harmonic. Multiplying h by $e^{i\theta}$ for some θ, we can safely assume that $h(z_0) = M$ is real; so if $h = u + iv$, then $u(z_0) = M$, $v(z_0) = 0$, so u attains its maximum and is therefore constant. Therefore $h = M + iv$ satisfies $|h(z)| = \sqrt{M^2 + v(z)^2} \leq M$ implies $|v(z)|^2 \leq 0$, i.e. $v(z) = 0$ for all $z \in D$, so h is constant.

Finally, we can extend to the boundary to get the maximum principle: if h is a complex-valued harmonic function on a bounded domain D such that h extends continuously to ∂D

and $|h(z)| \leq M$ for all $z \in \partial D$, then $|h(z)| \leq M$ for all $z \in D$. That is: a harmonic function can never be higher on the interior of such a domain then on its boundary.

To see this, observe that $D \cup \partial D$ is a closed and bounded set, hence compact, so f attains its maximum somewhere on this set. If it attained it on D, by the above principles it would be constant, hence also constant on ∂D with the same value, so the principle trivially holds. Otherwise, it attains its maximum on ∂D, so the principle again holds.