Lecture 7: fractional linear transformations

Complex analysis, lecture 4
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The last thing we want to mention in this section is the notion of fractional linear transformations, sometimes called Möbius transformations. We will spend only a little time on this idea; it is fundamental in some applications of complex analysis to number theory and certain kinds of geometry, and provides useful examples of conformal maps, but is otherwise not of fundamental importance.

Nevertheless, let's give a quick definition. A fractional linear transformation is a map of the form

$$f(z) = \frac{az+b}{cz+d}$$

where a, b, c, d are complex numbers. To make this well-defined, we should require that at the least c and d are not both zero. In fact, f will turn out to almost always be a bijection; if we want this to hold, we saw before that we need f' to be nonvanishing, and we can compute

$$f'(z) = \frac{ad-bc}{(cz+d)^2},$$

so we require $ad - bc \neq 0$.

You recognize the expression $ad - bc$: this is the determinant of a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which is zero if and only if the matrix is invertible. So we think of fractional linear transformations as somehow associated to invertible matrices; more precisely, for the algebraists, they give an action of the group of invertible matrices on the complex plane.

The careful listener will note that statements like the above are imprecise: f is not actually even a function $\mathbb{C} \to \mathbb{C}$ in general, since at $z = -d/c$ it is undefined. It is cleaner to define f on the Riemann sphere $\mathbb{C} \cup \{\infty\}$: on $z \neq \infty$, we define f by the formula above, and set $f(-d/c) = \infty$; and we write

$$f(\infty) = \lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{a+b/z}{c+d/z}\frac{a}{c}.$$

Note that therefore $f(\infty) = \infty$ if and only if $c = 0$.

Once we have extended to the Riemann sphere and required $ad - bc \neq 0$, we can show that every fractional linear transformation is invertible:

$$f^{-1}(z) = \frac{-dz+b}{cz-a},$$

another fractional linear transformation. (This is the fractional linear transformation associated to the inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.) One can likewise check that the composition of two such transformations is again a fractional linear transformation, but I'll leave the algebra as an exercise. The composition here corresponds to matrix multiplication.

Since f is analytic wherever it is defined (as a rational function), it is also conformal; by this extension to the Riemann sphere, once we require $ad - bc \neq 0$ so that f is invertible as above, this gives a conformal self-mapping of the Riemann sphere. By restricting a, b, c, d, we can get other self-mappings. For example, if a, b, c, d are all real, then f preserves the extended real line, and therefore also preserves its complement $^{\pm} = \mathbb{C} \setminus \mathbb{R}$. Both of these actions, especially the latter, turn out to be quite important in number theory, for reasons I won't get into now but am happy to discuss after class.

Let's now give some examples. If $c = 0$ and $d = 1$ (or after dividing a and b by d), we have $f(z) = az + b$, a familiar map (sometimes called an affine transformation). In particular fractional linear transformations include the scalings (or "dilations") $f(z) = az$ and translations $f(z) = z + b$. They also include the inversion $f(z) = \frac{1}{z}$, corresponding to $a = d = 0$ and $b = c = 1$. It turns out that every fractional linear transformation can be written as a composition of these three kinds of operations. For example,

$$f(z) = \frac{2z}{z+3}$$

can be written as follows: let $f_1(z) = z + 3$ (translation), $f_2(z) = \frac{1}{z}$ (inversion), $f_3(z) = -6z$ (scaling), and $f_4(z) = z + 2$ (another translation). Then

$$f_4(f_3(f_2(f_1(z)))) = 2 - \frac{6}{z+3} = \frac{2z}{z+3} = f(z).$$

A final point of interest I want to mention is the following. Note that not every quadruple (a, b, c, d) gives rise to a distinct fractional linear transformation: for example, if $a = b = c = d$, then the associated transformation is

$$f(z) = \frac{az + a}{az + a} = 1.$$

Of course, this is not strictly allowed since $ad - bc = 0$, but a tuple like $(a, 0, 0, a)$ is allowed, for $a \neq 0$, and would give

$$f(z) = \frac{az + 0}{0z + a} = z,$$

independent of a; so a whole one-dimensional space of tuples gives the same transformation. More generally, for any given tuple we could scale every parameter by the same constant and get the same transformation.

This suggests that the space of fractional linear transformations is in some sense at most three-dimensional. In fact, it is exactly three-dimensional, in the following sense: given any three distinct points $z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}$ and another triple $w_1, w_2, w_3 \in \mathbb{C} \cup \{\infty\}$, there is a unique fractional linear transformation f such that $f(z_1) = w_1$, $f(z_2) = w_2$, and $f(z_3) = w_3$.

We will not show this; the interested audience member is free to treat it as an exercise, or to discuss it with me later. For now, we simply observe that this means we can conformally map the Riemann sphere to itself in a large number of ways, with a relatively high degree of freedom (namely three degrees). For a simple example, if we want to send the tuple $(0, 1, \infty)$ to $(\infty, 1, 0)$, the corresponding transformation is

$$f(z) = \frac{1}{z},$$

which by the claim is the unique such transformation. It does not correspond to a unique (a, b, c, d), as above; we could use anything of the form $(0, a, a, 0)$.