Lecture 6: harmonic functions and conformal mappings

Complex analysis, lecture 4
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1. Harmonic functions

We begin with an important problem from analysis, apparently unrelated to complex analysis: the differential equation
$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$$
for a function $u(x_1, \ldots, x_n)$. This is Laplace's equation; the operator $\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is called the Laplacian Δ, so we could more simply rewrite the equation as
$$\Delta u = 0.$$
Solutions u to this differential equation are called harmonic functions. They have many applications both within pure mathematics and to numerous other fields.

We will be interested in the case $n = 2$, so
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$
We claim that these harmonic functions of two variables have a close relation to analytic functions, via the Cauchy–Riemann equations:

Proposition. *If $f = u+iv$ is an analytic function on a domain D, then viewing $D \subset \mathbb{C} \simeq \mathbb{R}^2$ as a subset of the plane, u and v are harmonic functions on D.*

This is a corollary of the Cauchy–Riemann equations: we have
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$
so
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$$
and therefore
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
i.e. u is harmonic. A similar argument works for v.

Given a harmonic function u on a domain D, it is then natural to ask if it "comes from" an analytic function f on D, i.e. whether there exists another harmonic function v on D such that $u + iv$ is an analytic function, and if so whether such an f is unique. (You could also swap the places of u and v, but this amounts to multiplying by i and so is equivalent.)

Such a function v is called a harmonic conjugate of u, so this is equivalent to asking whether u has a harmonic conjugate on D.

It turns out that it's not too hard to show uniqueness, up to an additive constant: that is, if v_1 and v_2 are both harmonic conjugates of u, then $v_1 - v_2$ is constant. Indeed, $f_1 = u+iv_1$ and $f_2 = u+iv_2$ are, by assumption, both analytic functions, so so is $(f_1-f_2)/i = ((u+iv_1)-(u+iv_2))/i = v_1 - v_2$. However, since v_1 and v_2 are real-valued, so is $v_1 - v_2$, and we saw last time that a real-valued analytic function must be constant.

The existence of a harmonic conjugate is more subtle. Let's try an example: let $u(x,y) = xy$. First, we claim that this is a harmonic function. Indeed, both second partial derivatives vanish, so $\Delta u = 0$ everywhere. How can we find a harmonic conjugate for u?

Well, we should solve the Cauchy–Riemann equations:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -x, \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = y.$$

The first equation tells us that $v(x,y) = -\frac{x^2}{2} + f(y)$ and the second that $v(x,y) = \frac{y^2}{2} + g(x)$, so equating these and setting $x = 0$ gives $f(y) = \frac{y^2}{2} + g(0)$ while setting $y = 0$ gives $g(x) = -\frac{x^2}{2} + f(0)$. Letting $C = f(0)$, we have $g(0) = C$, so

$$v(x,y) = -\frac{x^2}{2} + \frac{y^2}{2} + C$$

for some constant C. This gives the analytic function

$$f(z) = f(x+iy) = u(x,y)+iv(x,y) = xy+\frac{1}{2}(-x^2+y^2)i+Ci = -\frac{i}{2}(x+iy)^2+Ci = -\frac{i}{2}z^2+Ci,$$

which is visibly analytic in z; this is in fact already guaranteed by our construction of v, but it's good to check. So $v(x,y) = \frac{1}{2}(-x^2+y^2) + C$ is a harmonic conjugate of u for any constant C; and by the uniqueness above, these are the only harmonic conjugates of u.

This reasoning might lead you to believe that we can always find a harmonic conjugate. However, this is not true: it was implicitly important in the above that we were working on the entire complex plane $\mathbb{C}$. If we had a different domain, such as $D = \mathbb{C}\setminus\{0\} \simeq \mathbb{R}^2\setminus\{(0,0)\}$, this can fail. Consider for example $u(x,y) = \log(x^2+y^2)$, defined on this domain D. We have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2\frac{y^2-x^2}{(x^2+y^2)^2} + 2\frac{x^2-y^2}{(x^2+y^2)^2} = 0,$$

so u is harmonic on D. If v is a harmonic conjugate of u, applying the same method we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\frac{2y}{x^2+y^2}, \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}.$$

We could integrate as before, and we would find $v(x,y) = 2\tan^{-1}(y/x) = 2\arg(x+iy)$. But we know very well that there is no way to choose a function $\arg(z)$ which is continuous everywhere in $D = \mathbb{C}\setminus\{0\}$: we had to choose a branch cut. For v to be harmonic, it must be continuous, so there is no harmonic conjugate to u on D.

Nevertheless, under certain conditions on D we can show that the above method will always work: for example if D is the entire complex plane, an open disk, or a rectangle, or more generally a star-shaped domain. More precisely, the argument can only fail if it is possible to draw a path in D which contains inside of it a point which is not in D (in the example above, this would have been the origin). If D does not have any "holes," this is impossible (the precise version is: if D is simply connected) and so the argument works.

We'll come back to the maximally general version later. For the moment, let's assume we have a star-shaped domain D, with respect to some point (x_0, y_0). For any point (x, y) in D, we can choose a path connecting (x_0, y_0) and (x, y), and integrate along it it. The equation

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

gives

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial u}{\partial y}(t,s)dt + \frac{\partial u}{\partial x}(t,s)ds + C$$

for some constant C; in particular a harmonic conjugate exists. For the case $u(x, y) = xy$ as above, this gives

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -t\,dt + s\,ds + C = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x_0^2}{2} - \frac{y_0^2}{2} + C,$$

recovering the same harmonic conjugate as above after absorbing the extra additive constants into C.

2. Conformal mappings

Given a complex-valued function on a domain D, we want to think of it as a transformation between two spaces; thinking geometrically, an important property is that of preserving angles, which it turns out is satisfied by many functions of interest. In order to think more deeply about this we need to say what this means.

Consider a map $\gamma : [0, 1] \to \mathbb{C}$, given by $\gamma(t) = x(t) + iy(t)$ for real-valued functions x and y. If x and y are differentiable, we say that γ is a smooth curve in $\mathbb{C}$, with an endpoint at $z_0 = \gamma(0)$. In particular we can take the derivative $\gamma'(0) = x'(0) + iy'(0)$ to get a complex number approximating the curve near z_0: this is called the tangent vector to the curve γ at z_0, and is a complex incarnation of the usual tangent vector. (Note in fact we only need γ differentiable at zero to make this definition.)

Given two curves γ_1, γ_2 both terminating at z_0, we say that the angle between these curves (at z_0) is the angle between their tangent vectors at z_0, so long as both tangent vectors are nonzero.

We would like to understand how these curves interact with complex functions. If γ is a curve terminating at z_0 and $f : D \to \mathbb{C}$ is some complex function on a domain containing z_0 and further containing the image of γ, then $f \circ \gamma : [0, 1] \to \mathbb{C}$ defines a new curve in the complex plane. If f is differentiable (or at least at $z_0 = \gamma(0)$), $f \circ \gamma$ is even again a smooth curve (or respectively at least we can define the tangent vector $(f \circ \gamma)'(0)$).

We can now define a conformal map: if $f : D \to \mathbb{C}$ is a complex function, we say it is conformal at $z_0 \in D$ if for every pair of curves $\gamma_1, \gamma_2 : [0, 1] \to D$ in D with $\gamma_1(0) = \gamma_2(0) = z_0$ which are differentiable at zero with nonzero tangent vectors $\gamma_1'(0), \gamma_2'(0)$, then the curves $f \circ \gamma_1$ and $f \circ \gamma_2$ are both differentiable at 0 with nonzero tangent vectors and the angle between $(f \circ \gamma_1)'(0)$ and $(f \circ \gamma_2)'(0)$ is the same as the angle between $\gamma_1'(0)$ and $\gamma_2'(0)$. More informally, f preserves angles at z_0.

We say that a function $f : D \to D'$, where D and D' are domains in $\mathbb{C}$, is a conformal mapping if it is conformal at each point of D and it is a bijection.

For example, $f(z) = z + c$ for some constant c certainly preserves angles; the same is true for multiplication by some constant, $f(z) = az$ (indeed, then $f \circ \gamma = a\gamma$, and multiplication by $a = re^{i\theta}$ is a combination of scaling by r and rotating by θ, hence doesn't change the angle between two vectors when both are scaled/rotated). On the other hand, $f(z) = \overline{z}$ flips angles, so it is not conformal.

This might remind us of analytic functions. Indeed, one way to guarantee that $f \circ \gamma$ is differentiable at 0 (if γ is) was to require that f be differentiable at z_0. Under this assumption (as well as continuity of the derivative, so f is analytic at z_0), we can actually directly compute the tangent vector of $f \circ \gamma$: by the chain rule,

$$\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t))\gamma'(t),$$

so at $t = 0$ we find

$$(f \circ \gamma)'(0) = f'(z_0)\gamma'(0).$$

In other words, the tangent vector of $f \circ \gamma$ at $f(z_0)$ is just the product of the tangent vector of γ at z_0 with $f'(z_0)$.

Thus for two different curves γ_1, γ_2, their tangent vectors are both multiplied by the same factor under composition by f, so as above the angle is preserved provided $f'(z_0) \neq 0$. Thus we have proven that if f is analytic at z_0 with nonzero derivative, it is also conformal at z_0.

For example, $f(z) = z^n$ for n a positive integer is *not* conformal at 0, since it multiplies angles around the origin by n. This makes sense: $f'(0) = n \cdot 0^{n-1} = 0$. However, for $z \neq 0$ the above result shows that f is conformal at z.

One can also show that conformal mappings are analytic, but we put this off for now.

The condition that a conformal mapping be a bijection avoids some possible issues with self-intersections of curves, but more fundamentally it suggests that conformal mappings should be thought of as equivalences: domains in the complex numbers are equipped with a notion of angles between tangent vectors as above, and conformal mappings are equivalences of spaces with this structure. We will not be especially precise about this, but one can often think of two domains with a conformal mapping between them as somehow the same. One of the capstone results we will (hopefully) see towards the end of the class is a classification of the simply connected domains in the complex numbers or, equivalently, in the Riemann sphere, considered up to conformal equivalence; it is interesting that, although this is a purely geometric/topological statement, the tools involved in its proof are complex-analytic in nature.

We conclude with some examples. Consider the domain $D = \{z \neq 0 : -\frac{\pi}{2} < \operatorname{Arg} z < \frac{\pi}{2}\} = \{z : \operatorname{Re} z > 0\}$. We know from our study of the square root function that $f(z) = z^2$ gives a bijection from D to $D' = \{z \neq 0 : \operatorname{Arg} z \neq \pi\} = \mathbb{C} \setminus (-\infty, 0]$. Since 0 is not in either D or D', we saw before that f is conformal at every point of D, so it is a conformal mapping $D \to D'$. If we restricted to a sector like $\{0 < \arg z < \theta\}$ for $0 < \theta \leq \pi/2$, f would give a conformal mapping from this domain to $\{0 < \arg z < 2\theta\}$.

Similarly, $f(z) = \sqrt{z}$, after taking the principal branch, would give a conformal mapping $D' \to D$; after restricting e.g. to the upper half-plane $D \subset D'$, f maps D conformally onto the region $\{-\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$.

The function $f(z) = e^z$ is analytic everywhere, with derivative everywhere nonzero, so it is conformal everywhere. However, it is not a conformal mapping $\mathbb{C} \to \mathbb{C}$, or even $\mathbb{C} \to \mathbb{C} \setminus \{0\}$ onto its image, because it is not injective. If we restrict to the strip $\{z : -\pi < \operatorname{Im} z < \pi\}$, we do get a conformal mapping f from this strip to $\mathbb{C} \setminus (-\infty, 0]$, where we have to make the branch cut on the target so that the inverse, the principal branch of the logarithm, is well-defined, just like for the square and square root above.