Lecture 5: the Cauchy–Riemann equations

Complex analysis, lecture 4
September 10, 2025

Last time, we introduced analytic (or holomorphic) functions on a domain as those whose complex derivative is well-defined and continuous at every point in the domain. Today, we will find an explicit way to check, given a complex-valued function, if it is analytic or not.

We work in Cartesian coordinates both on the input and on the output: write $z = x+iy$, and $f(z) = u + iv$. Thinking of f as depending on the two real numbers x and y, we can think of each of u and v as real-valued functions of x and y:

$$f(x + iy) = u(x, y) + iv(x, y).$$

Let's study the derivative in this setting, using the formula

$$f'(z) = \lim_{h\to 0} \frac{f(z + h) - f(z)}{h}.$$

Here the limit is in the complex sense, so we must be able to take h approach zero from any direction. We will study the cases when h is approaching along the real or imaginary line.

Suppose h is real. Then

$$\begin{aligned}\frac{f(z + h) - f(z)}{h} &= \frac{f(x + h + iy) - f(x + iy)}{h}\\ &= \frac{u(x + h, y) + iv(x + h, y) - u(x, y) - iv(x, y)}{h}\\ &= \frac{u(x + h, y) - u(x, y)}{h} + \frac{v(x + h, y) - v(x, y)}{h} i.\end{aligned}$$

As $h \to 0$ along the real line, the limit—if it exists—is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i.$$

Let's now take the limit along the imaginary axis; for clarity, we'll still take h to be a real number, and change z by ih. We have

$$\begin{aligned}\frac{f(z + ih) - f(z)}{ih} &= \frac{f(x + i(y + h)) - f(x + iy)}{ih}\\ &= \frac{u(x, y + h) + iv(x, y + h) - u(x, y) - iv(x, y)}{ih}\\ &= -\frac{u(x, y + h) - u(x, y)}{h} i + \frac{v(x, y + h) - v(x, y)}{h}.\end{aligned}$$

Taking the limit as $h \to 0$, the right-hand side becomes

$$-\frac{\partial u}{\partial y} i + \frac{\partial v}{\partial y}.$$

Now, if f is going to be differentiable at z, these limits must both exist and must both agree. So if f is differentiable at z, both first-order partial derivatives of u and v must exist; and, equating the expressions above and taking real and imaginary parts, we must have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are called the Cauchy–Riemann equations.

Theorem. *Let $f = u+iv$ be a complex-valued function on a domain D. Then f is analytic on D if and only if u and v have first-order partial derivatives defined and continuous everywhere on D which satisfy the Cauchy–Riemann equations.*

We have already essentially shown one direction of this result: if f is differentiable at z, then it satisfies the Cauchy–Riemann equations at z. (To replace "differentiable" with "analytic," we only need to add the requirement that the partial derivatives be continuous.) What remains is to prove the converse: if u and v have partial derivatives satisfying the Cauchy–Riemann equations, then f is continuous.

We show this using first-order approximation: if $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are defined near (x, y), then for small real numbers j, k we have

$$u(x + j, y + k) = u(x, y) + \frac{\partial u}{\partial x} \cdot j + \frac{\partial u}{\partial y} \cdot k + R(j, k)$$

where R is some function such that $\lim_{j \to 0} \lim_{k \to 0} \frac{R(j,k)}{|(j,k)|} = 0$. The same formula holds for v in place of u; write $S(j, k)$ for the remainder term in place of R. Then we can compute

$$\begin{aligned} \frac{f(z + h) - f(z)}{h} &= \frac{f(x + j + i(y + k)}{j + ik} \\ &= \frac{u(x + j, y + k) + iv(x + j, y + k) - u(x, y) - iv(x, y)}{j + ik} \\ &= \frac{\frac{\partial u}{\partial x} \cdot j + \frac{\partial u}{\partial y} \cdot k + R(j, k) + i\left(\frac{\partial v}{\partial x} \cdot j + \frac{\partial v}{\partial y} \cdot k + S(j, k)\right)}{j + ik}. \end{aligned}$$

If we assume that the Cauchy–Riemann equations hold, so that $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, we can rewrite this as

$$\frac{\frac{\partial u}{\partial x}(j + ik) + \frac{\partial u}{\partial y}(k - ij) + R(j, k) + iS(j, k)}{j + ik} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} i + \frac{R(j, k) + iS(j, k)}{j + ik}.$$

Taking the limit as $j, k \to 0$, the third term we know tends to zero in absolute value, so we can drop it, and we're left with the first two terms which by assumption are well-defined and continuous on D. Hence f is in fact analytic on D under these assumptions.

We check some examples. For $f(z) = z = x + iy$, we have $u(x, y) = x$ and $v(x, y) = y$, so $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$, and $\frac{\partial v}{\partial y} = 1$, so the equations hold by inspection.

A slightly more complicated example is $f(z) = e^z$. Writing $z = x + iy$, this is $e^x \cos y + ie^x \sin y$ by Euler's formula, so $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. We find

$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial u}{\partial y} = -e^x \sin y, \qquad \frac{\partial v}{\partial x} = e^x \sin y, \qquad \frac{\partial v}{\partial y} = e^x \cos y,$$

so again the equations hold.

Since the equations are linear, linear combinations of analytic functions are analytic (which we already know by rules of differentiation). A more important property is the following, reflecting a basic principle of integral calculus.

Proposition. *If $f(z)$ is analytic on a domain D and $f'(z) = 0$ for all $z \in D$, then f is constant on D.*

Indeed, if f is analytic then it satisfies the Cauchy–Riemann equations, and we computed above that then

$$f'(z) = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} i = 0,$$

so taking real and imaginary parts we see that u has both partial derivatives everywhere zero, hence u is constant. By the Cauchy–Riemann equations, both partial derivatives of v are also zero, so v is also constant, hence f is constant.

One can also check other analogous properties. For example, if f is real-valued and analytic on a domain D, then it must be constant. This might be surprising, since we're used to restricting to the real line (or a subset of it) to recover a real-valued function; but note that no (nonempty) subset of the real line is a domain! Indeed, using the Cauchy–Riemann equations, if $v = 0$ then its partial derivatives both vanish, so so do those of u, so u must be constant, so $f = u + 0$ is constant. This immediately tells us that for example $f(z) = |z|^2 = x^2 + y^2$ is not an analytic function; this can be verified from the Cauchy–Riemann equation.

Note again that from the apparently weak condition that a function be complex-differentiable, we've deduced that it must in fact satisfy a pair of differential equations, a much stronger-looking condition! This relates back to our slogan from last time: the existence (and conditions on the behavior) of complex limits is a much stronger assumption than it looks.

In everything above, we used Cartesian coordinates. However, we could just as well have used polar coordinates: writing $z = re^{i\theta}$ and $f(z) = u + iv$ with u and v viewed as functions of r and θ, by varying either r or θ, one can derive the polar form of the Cauchy–Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \qquad \frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}.$$

A slightly different application of the Cauchy–Riemann equations is to showing that inverse functions of analytic functions are analytic, at least after restricting to some neighborhood of a given point (if nothing else to ensure that they are single-valued). Viewing $D \subset \mathbb{C} \simeq \mathbb{R}^2$ and $f : D \to \mathbb{C} \simeq \mathbb{R}^2$ as a function from a region in the plane to the plane, from multivariable calculus the invertibility of f is determined by the Jacobian matrix

$$J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

In particular, f is invertible when J_f is, equivalently when $\det J_f$ is nonzero. Using the Cauchy–Riemann equations to substitute for the partial derivatives in y, we can write the determinant as

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial f}{\partial x}\right|^2.$$

Since f is analytic, the partial derivative with respect to x is just f', so this determinant is just $|f'(z)|^2$.

Therefore if $f'(z) \neq 0$, then f is locally invertible at z. The standard formula from calculus tells us that the derivative of f^{-1} at $f(z)$ is $\frac{1}{f'(z)}$, so since f' is nonzero and continuous at z by analyticity, it follows that f^{-1} is also analytic at z.

For example, the principal branch $\operatorname{Log} z$ of the logarithm is analytic away from the branch cut at $\theta = \pi$ and has derivative at $z = e^w$ given by $\frac{1}{e^w} = \frac{1}{z}$. Since any other branch of the logarithm differs from $\operatorname{Log} z$ by a constant, their derivatives are the same, so any branch of the logarithm has derivative $\frac{1}{z}$, which is well-defined and continuous away from $z = 0$.