Lecture 27: harmonic functions and the Poisson integral formula

Complex analysis, lecture 4
November 24, 2025

1. The Poisson integral formula

In the last few lectures, we have returned to conformal functions from earlier in the semester, and greatly strengthened what we knew how to say about them earlier. We now want to try to do something similar with harmonic functions. In particular, we saw several units ago that harmonic functions satisfy the mean value property; our main result will be that this actually completely characterizes harmonic functions, just as we first proved that analytic maps with nonvanishing derivatives are conformal, and only later proved that this is a complete characterization.

Our central tool will be the Poisson integral formula, an analogue for harmonic functions of the Cauchy integral formula. We phrase this in terms of solving the Dirichlet problem on the disk.

What does this mean? Let's work on the unit disk $\mathbb{D}$ for simplicity. Suppose we have a continuous function $h : \partial\mathbb{D} \to \mathbb{C}$, i.e. a function $h(e^{i\theta})$. The Dirichlet problem is the problem of extending h to a harmonic function $\tilde{h}$ on $\mathbb{D}$.

Any solution to the Dirichlet problem on the disk is unique: if we have two solutions $\tilde{h}_1, \tilde{h}_2$, then $\tilde{h}_1 - \tilde{h}_2$ is harmonic on $\mathbb{D}$, and on $\partial\mathbb{D}$ we have $\tilde{h}_1(e^{i\theta}) - \tilde{h}_2(e^{i\theta}) = h(e^{i\theta}) - h(e^{i\theta}) = 0$, so by the maximum principle $\tilde{h}_1 - \tilde{h}_2 = 0$ on all of $\mathbb{D}$, so $\tilde{h}_1 = \tilde{h}_2$. Therefore it suffices to find a formula for $\tilde{h}$ on $\mathbb{D}$; if it's harmonic and recovers the right values on the boundary, it must be the only solution.

The idea is to start with the case where $h(e^{i\theta})$ is a polynomial in $e^{i\theta}$; we'll see that the resulting formula generalizes.

The simplest case is that of a monomial, $h(e^{i\theta}) = h(e^{in\theta})$ for some nonnegative integer n. In this case, a harmonic extension to $\mathbb{D}$ is $\tilde{h}(z) = \tilde{h}(re^{i\theta}) = r^n e^{in\theta} = z^n$. For negative n, z^n is no longer continuous on $\mathbb{D}$, but note that on the boundary, $z^{-1} = \overline{z}$, so we could instead take the extension of this formula, or $\tilde{h}(z) = \tilde{h}(re^{i\theta}) = r^{|n|}e^{in\theta}$, which for $n < 0$ gives $\overline{z}^n$. This is not analytic, but it is conjugate-analytic, and so is also harmonic. (Recall that being analytic is equivalent to $\frac{\partial}{\partial \overline{z}} f = 0$ while being conjugate-analytic is equivalent to $\frac{\partial}{\partial z} f = 0$, and being harmonic is equivalent to $\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f = 0$, so analytic and conjugate-analytic functions are both harmonic.)

Thus for a trigonometric polynomial

$$h(e^{i\theta}) = \sum_{n=-N}^{N} a_n e^{in\theta},$$

a harmonic extension to $\mathbb{D}$ (and therefore the only such extension) is given by

$$\tilde{h}(z) = \tilde{h}(re^{i\theta}) = \sum_{n=-N}^{N} a_n r^{|n|} e^{in\theta}.$$

We would like a way to recover the coefficients a_n from the function $h(e^{i\theta})$. Integrating,

$$\int_0^{2\pi} h(e^{i\theta})\,d\theta = \int_0^{2\pi} \sum_{n=-N}^{N} a_n e^{in\theta}\,d\theta = \sum_{n=-N}^{N} a_n \int_0^{2\pi} e^{in\theta}\,d\theta = 2\pi a_0,$$

since the integral is 0 unless $n = 0$, in which case it is 2π. If we were to multiply h by $e^{-im\theta}$, this would give $2\pi a_m$ instead, so in general

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) e^{-in\theta}\,d\theta.$$

Note that this is true for all n, not just $|n| \le N$; if $|n| > N$, it should give 0. Therefore in our sums we can freely take the indices from $-\infty$ to ∞ without changing anything.

If we return to our formula for $\tilde{h}$ on $\mathbb{D}$, we then get

$$\begin{aligned}\tilde{h}(z) = \tilde{h}(re^{i\theta}) &= \sum_{n=-\infty}^{\infty} e^{in\theta} r^{|n|} \cdot \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\phi}) e^{-in\phi}\,d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\phi}) \sum_{n=-\infty}^{\infty} e^{in(\theta-\phi)} r^{|n|}\,d\phi.\end{aligned}$$

This motivates us to give a name to the inner sum: we set

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} r^{|n|},$$

which converges for $|r| < 1$ (uniformly on any closed disk of radius < 1). This is called the Poisson kernel. Then the above formula reads

$$\tilde{h}(z) = \tilde{h}(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\phi}) P_r(\theta - \phi)\,d\phi.$$

Making a substitution and using the fact that $h(e^{i\phi})$ is periodic with period 2π, we could rewrite this as

$$\tilde{h}(z) = \tilde{h}(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i(\theta-\phi)}) P_r(\phi)\,d\phi.$$

Our next goal is to rewrite the Poisson kernel in a simpler form. To deal with the absolute value, we split the sum up into $n > 0$, $n = 0$, and $n < 0$:

$$\begin{aligned}P_r(\theta) &= \sum_{n=-\infty}^{-1} e^{in\theta} r^{|n|} + 1 + \sum_{n=1}^{\infty} e^{in\theta} r^{|n|} \\ &= \sum_{n=1}^{\infty} (re^{-i\theta})^n + 1 + \sum_{n=1}^{\infty} (re^{i\theta})^n \\ &= \frac{\overline{z}}{1 - \overline{z}} + 1 + \frac{z}{1 - z}\end{aligned}$$

where $z = re^{i\theta} \in \mathbb{D}$. Simplifying, this is

$$P_r(\theta) = \frac{1-|z|^2}{|1-z|^2} = \frac{1-r^2}{(1-re^{i\theta})(1-re^{-i\theta})} = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

This immediately tells us for example that $P_r(\theta) > 0$ for all r, θ. It is a decreasing function for $0 < \theta < \pi$ and increasing for $\pi < \theta < 2\pi$, and satisfies $P_r(2\pi - \theta) = P_r(\theta)$. Since $P_r(\theta) \to 0$ as $r \to 1$ and $P_r(\theta)$ is decreasing on $[\theta, \pi)$, for any fixed $\delta > 0$, $P_r(\theta) \to 0$ uniformly on $\theta \in [\delta, \pi]$.

(It is sometimes convenient to work with $[-\pi, \pi]$ instead of $[0, 2\pi]$, in which case everything is as above, but we can state the previous claim in terms of $|\theta|$, and $P_r(-\theta) = P_r(\theta)$ instead of $P_r(2\pi - \theta)$.)

Finally,

$$\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta)\, d\theta = 1$$

for all r. This is not immediate from the definition, but follows from the formula above for $h(e^{i\theta}) = 1$, since $\widetilde{h}(z) = 1$ on $\mathbb{D}$ is a harmonic extension to $\mathbb{D}$ (and therefore the only one).

For $r = 0$, the Poisson kernel is the constant function 1. As $r \to 1$, it becomes less and less flat, approaching a spike at $\theta = 0$. Since the integral of $P_r(\theta)$ is constant in r, the total area bounded by the curve remains constant, so since for θ far from 0 (or 2π), so $\cos\theta$ small, the values go to 0 as $r \to 1$, the value at 0 must spike to infinity. This is a version of the Dirac delta function, which some of you may be familiar with.

Now, we only know that the formula $\widetilde{h}(z)$ above is a solution to the Dirichlet problem when h is a trigonometric polynomial. However, the formula makes sense for any continuous function h on $\partial\mathbb{D}$. We define this to be the Poisson integral of h:

$$\tilde{h}(z) = \tilde{h}(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i(\theta-\phi)})P_r(\phi)\, d\phi = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\phi})P_r(\theta-\phi)\, d\phi.$$

If we would like to avoid the explicit dependence on polar coordinates, by the discussion above we can rewrite the Poisson kernel $P_r(\phi)$ in terms of $w = re^{i\phi}$ as $\frac{1-|w|^2}{|1-w|^2}$, with $d\phi = \frac{dw}{iw}$, so this is

$$\tilde{h}(z) = \frac{1}{2\pi i}\int_{|w|=|z|} h(z/w) \cdot \frac{1-|w|^2}{|1-w|^2}\, dw.$$

The previous form however is usually more useful.

Theorem. *Let $h(e^{i\theta})$ be a continuous function on $\partial\mathbb{D}$. Then its Poisson integral $\tilde{h}(z)$ is a harmonic function on $\mathbb{D}$ with boundary values h, i.e. $\lim_{z\to\zeta}\tilde{h}(z) = h(\zeta)$ for $\zeta = e^{i\theta} \in \mathbb{D}$.*

In particular, $\tilde{h}$ solves the Dirichlet problem on $\mathbb{D}$.

There are two pieces to the theorem: first, that $\tilde{h}$ is harmonic, which is a direct calculation of the partial derivatives by differentiating under the integral sign (this is slightly annoying because it requires either working in polar coordinates or converting to Cartesian, but is not too bad); and second, that the boundary values are in fact h. This can be proven concretely using some of the properties of the Poisson kernel listed above, but a more conceptual proof goes back to the trigonometric polynomial case. There, this was true essentially by construction, so this follows from the statement that all continuous functions h on $\partial\mathbb{D}$ can be approximated in a precise sense by trigonometric polynomials. This is very closely related to the idea of Fourier series: any such function can be written as a series

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

for some coefficients a_n, generalizing trigonometric polynomials in exactly the way that Laurent series generalize usual polynomials.

A consequence is the following analogue of Cauchy's formula on the disk, sometimes by analogy called the Poisson integral formula.

Theorem. *Let $h : \mathbb{D} \to \mathbb{C}$ be a harmonic function extending continuously to $\partial\mathbb{D}$. Then for any $z_0 = re^{i\theta} \in \mathbb{D}$,*

$$h(z_0) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i(\theta-\phi)}) P_r(\phi)\, d\phi.$$

That is, we can recover the values of a harmonic function on the disk from its values on the boundary, just as we could recover the values of an analytic function, albeit by a different-looking formula. Indeed, the given integral is the Poisson integral of the boundary values for h, so it gives a harmonic extension to $\mathbb{D}$; since h itself is such an extension and these are unique, the claim follows.

2. Characterizing harmonic functions

The Poisson integral formula is interesting and powerful, in that it lets us apply tools to harmonic functions otherwise only available for analytic functions such as recovering a function on a domain from its values on the boundary. However, it is also very specialized: it holds only on the unit disk $\mathbb{D}$ centered at the origin.

It is not too difficult to generalize this to other disks, by scaling and translation. In other words, for any disk $D \subset \mathbb{C}$, we can solve the Dirichlet problem on D: given a continuous function $h : \partial D \to \mathbb{C}$, there is a unique harmonic function $\tilde{h} : D \to \mathbb{C}$ with boundary values h, which we can write explicitly via an integral formula.

While this is a simple consequence of the Poisson integral formula it lets us fully characterize harmonic functions on *any* domain Ω.

Recall that a harmonic function h on Ω satisfies the mean value property: for any $z_0 \in \Omega$, on any sufficiently small disk $D \subset \Omega$ of radius r centered at z_0 we have

$$h(z_0) = \frac{1}{2\pi} \int_0^{2\pi} h(z_0 + re^{i\theta})\, d\theta.$$

We claim that this property actually characterizes harmonic functions:

Theorem. *Let $h : \Omega \to \mathbb{C}$ be a continuous function. Then h satisfies the mean value property if and only if it is harmonic.*

Proof. We have seen that a harmonic function satisfies the mean value property, so it remains to see the converse: if h satisfies the mean value property, it is harmonic.

Suppose that h satisfies the mean value property. Fix $z_0 \in \Omega$, and let D be a sufficiently small disk of radius r around z_0. Restricting h to ∂D gives a continuous function $\partial D \to \mathbb{C}$, so since we can solve the Dirichlet problem on D there exists a unique harmonic function $g : D \to \mathbb{C}$ with boundary values h.

Since h itself is a continuous function on D with boundary values h, $h - g$ is a continuous function on D with boundary values 0. Since g is harmonic, it satisfies the mean value property, so since h does by hypothesis, so does $h - g$.

Now, recall that when we proved the maximum principle for harmonic functions, back in Lecture 9, we did it in three steps: first, we showed that for a harmonic function $u : D \to \mathbb{R}$, $\omega = -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy$ is closed; then we used this to show that u satisfies the mean value property; and finally we used the mean value property to prove the maximum principle. In particular, we actually don't need to know that u is harmonic to know that it satisfies the maximum principle: we only need to know that it satisfies the mean value property. (Of course, these are actually the same so there's no real distinction, but the difference is important for this proof!)

So since $h - g$ satisfies the mean value property, it satisfies the maximum principle as well. Since its boundary values are 0, it follows that $h - g$ must also vanish, i.e. $h = g$. Since g is harmonic, so is h, i.e. h is the unique extension of its boundary values to a harmonic function on D.

Since h is therefore harmonic on a neighborhood of every point $z_0 \in \Omega$, it is harmonic on Ω. □

This is remarkable in that we started with the assumption that h is continuous, and concluded that it is harmonic and therefore smooth, by assuming only that h satisfies the mean value property which a priori doesn't involve any differential information.

The characterization of harmonic functions by the mean value property can be thought of as the analogue to Morera's theorem: there, we check that the integrals over various rectangles inside a domain all vanish (i.e. are independent of the path), and here we are checking that the integrals over circles of different radii around a central point are equal, i.e. that the integral is independent of path in that all of these are equal to the integral over the "trivial" path $\gamma(t) = z_0$.

This concludes our regular material for the semester. After Thanksgiving, we will return for review, the final midterm, and possibly presentations. Happy Thanksgiving!