Lecture 26: the Riemann mapping theorem

Complex analysis, lecture 4
November 21, 2025

Although we stated it informally last time, let's begin by formally writing down the Riemann mapping theorem.

Theorem (Riemann mapping theorem)**.** *Every simply connected domain $D \subset \mathbb{C}$ is conformally equivalent to either $\mathbb{D}$ or $\mathbb{C}$.*

Last time, we saw that $\mathbb{D}$ and $\mathbb{C}$ are not conformally equivalent, and that the only simply connected domain conformally equivalent to $\mathbb{C}$ is $\mathbb{C}$ itself. So the theorem equivalently tells us that every simply connected strict subdomain of $\mathbb{C}$ is conformally equivalent to the unit disk $\mathbb{D}$.

One might ask what happens if we drop the hypothesis that our domain be simply connected. In this case, the statement would fail: if $f : \Omega \to \mathbb{D}$ is a conformal map and γ is a simple closed curve in Ω, since $f \circ \gamma$ is a simple closed curve in $\mathbb{D}$ we can deform it to a point, since $\mathbb{D}$ is simply connected. Hence by applying f^{-1} to these deformations we can also deform γ to a point, so Ω must also be simply connected: that is, if it is *not* simply connected, it is not conformally equivalent to $\mathbb{D}$.

Although it is possible to talk sensibly about classifying domains up to conformal equivalence without assuming they are simply connected, the classification is much wilder and harder to describe concretely. (For those with topology backgrounds, you might think of an arbitrary such domain—or more generally Riemann surface—as a quotient by a discrete group of its universal cover, which must be simply connected, so this amounts to a discussion of which discrete groups are allowed.)

Last time, we discussed conformal maps not only on $\mathbb{C}$ but also on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. This suggests a more general problem: if we allow our domains D to be contained in $\mathbb{C} \cup \{\infty\}$ rather than just in $\mathbb{C}$, how can we classify them? (For example, $\{\infty\} \cup \{z : |\operatorname{Re}(z)| > 1\}$ is a domain in the Riemann sphere, although $\{z : |\operatorname{Re}(z)| > 1\}$ is *not* a domain in $\mathbb{C}$ since it is disconnected. Once we add in the point at infinity though we can connect the two components by a line passing through ∞.)

Corollary. *Every simply connected domain $D \subset \mathbb{C} \cup \{\infty\}$ is conformally equivalent to either $\mathbb{D}$, $\mathbb{C}$, or $\mathbb{C} \cup \{\infty\}$.*

Proof. If Ω is not equal to the entire Riemann sphere $\mathbb{C} \cup \{\infty\}$, then there is some point $c \in (\mathbb{C} \cup \{\infty\}) \setminus \Omega$. We saw last time that a fractional linear transformation sending $c \mapsto \infty$, such as e.g. $z \mapsto \frac{cz}{z-c}$ (which also maps $\infty \mapsto c$), is a conformal map $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, and it sends Ω to a subdomain of $(\mathbb{C} \cup \{\infty\}) \setminus \{\infty\} = \mathbb{C}$. Therefore by the Riemann mapping theorem, if Ω is not the whole Riemann sphere it must be either $\mathbb{C}$ or conformally equivalent to $\mathbb{D}$. $\square$

Similarly to the previous case, the only subdomain of $\mathbb{C} \cup \{\infty\}$ conformally equivalent to the whole Riemann sphere is $\mathbb{C} \cup \{\infty\}$ itself: if there was a conformal map $f : \mathbb{C} \cup \{\infty\} \to \Omega$, its restriction to $\mathbb{C}$ would give a conformal map onto $\Omega \setminus \{f(\infty)\}$, so the latter is conformally equivalent to $\mathbb{C}$, hence we can identify it with $\mathbb{C}$ and therefore $\Omega = \mathbb{C} \cup \{\infty\}$.

Although the proof of the Riemann mapping theorem is beyond the scope of this class, we will attempt to give some sketch at the key ideas that go into the proof.

Let $\Omega \subset \mathbb{C}$ be a simply connected domain, not equal to all of $\mathbb{C}$. Our goal is to show that there exists a conformal map $\Omega \to \mathbb{D}$.

On all of $\mathbb{C}$, recall that the function $z \mapsto \sqrt{z}$, after picking a branch, has image given by a half-plane. This is promising since we know that any half-plane is conformally equivalent to $\mathbb{D}$ by an explicit construction from last time. However, the branch point is a problem: this is not an analytic function on $\mathbb{C}$, no matter how we pick the branch. More generally, $\sqrt{z-a}$ has a branch point at $a \in \mathbb{C}$.

However, since $\Omega \neq \mathbb{C}$, we can find $a \in \mathbb{C} \setminus \Omega$, so the function $z \mapsto \sqrt{z-a}$, after choosing any branch and restricting to Ω, has no branch point (in Ω!) and so is an analytic function on Ω. (To make this fully precise, one needs to use the fact that Ω is simply connected, so that one can choose an analytic branch of the complex logarithm on it.) Its derivative $\frac{1}{\sqrt{z-a}}$ is nowhere vanishing, so this gives a conformal function $\Omega \to \mathbb{C}$, which with a little more work we can see is an injection, with image some subdomain of the half-plane. Composing with the fixed conformal map from the half-plane to $\mathbb{D}$ shows that Ω is conformally equivalent to some subdomain of $\mathbb{D}$, or equivalently that there exists an injective conformal function $\Omega \to \mathbb{D}$; what remains is to show that it is surjective as well. Equivalently, we can assume that $\Omega \subset \mathbb{D}$, and we want to show that there is a conformal map $\Omega \to \mathbb{D}$ somehow "stretching" Ω to the full disk.

The idea will be to look at a *family* of injective conformal functions $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$ for some fixed point $z_0 \in \Omega$. Since these are injective, we can think of them as "stretching" Ω as desired; quantitatively, the amount of stretching can be measured by looking at $|f'(z_0)|$.

Since we have constructed an injection $\Omega \to \mathbb{D}$ with all the desired properties, our family of maps is nonempty. There are a collection of (difficult) theorems which together prove results to the effect that we can find an *extremal* element f of our family, in this case meaning with maximal value of $|f'(z_0)|$. Suppose f is not surjective, so we can find $a \in \mathbb{D}$ not in the image of f. By applying a conformal self-map of $\mathbb{D}$, as last time, we move a to 0. We can then apply the square root function to the image of this conformal map applied to f, which will be analytic as above after fixing a branch. Finally, we take another conformal self-map of $\mathbb{D}$ to move the image of z_0 under this composite back to 0. The result is again injective, conformal, and has image in $\mathbb{D}$, so we can take it to be an element of our family of functions as well. Writing out these maps, one can show that the derivative of the resulting map at z_0 is

$$-\frac{1+|a|}{2\sqrt{a}} f'(z_0),$$

so in particular since $|a| < 1$ we have $1 + |a| - 2\sqrt{|a|} = (1 - \sqrt{|a|})^2 > 0$, so $1 + |a| > 2\sqrt{|a|}$, and so this has absolute value strictly greater than $|f'(z_0)|$, contradicting the maximality of $|f'(z_0)|$. Therefore f must in fact be surjective, hence gives a conformal map $\Omega \to \mathbb{D}$.