Lecture 25: conformal maps to the disk

Complex analysis, lecture 4
November 19, 2025

We have mentioned before the philosophy that we consider domains with a conformal map between them to be equivalent. Our goal this week is to work up to the statement of the Riemann mapping theorem, which gives a full classification of all of the simply connected domains in $\mathbb{C}$ (or the Riemann sphere $\mathbb{C} \cup \{\infty\}$) up to conformal equivalence: every such domain has a conformal map to either $\mathbb{C} \cup \{\infty\}$ (if we're working in the Riemann sphere), $\mathbb{C}$ itself, or the unit disk $\mathbb{D}$.

We will not prove this theorem, nor formally state it quite yet, but we'll use it as motivation: if this theorem is true, then any domain in $\mathbb{C}$ other than $\mathbb{C}$ itself has a conformal map to the unit disk. Indeed, this follows from the following theorem:

Theorem (Picard's theorem)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is an analytic function, its image is either a single point $\{c\}$, the entire complex plane $\mathbb{C}$, or $\mathbb{C} - \{c\}$ for some $c \in \mathbb{C}$.*

Proof sketch. Let f be an analytic function $\mathbb{C} \to \mathbb{C}$ such that there exist $c_1 \neq c_2$ not in the image of f. For any fixed c_1, c_2, there is a holomorphic surjection $g : \mathbb{D} \to \mathbb{C} \setminus \{c_1, c_2\}$, whose construction is somewhat complicated and which we omit. Then one can show that $f = g \circ \widetilde{f}$ for some map $\widetilde{f} : \mathbb{C} \to \mathbb{D}$, and then $\widetilde{f} : \mathbb{C} \to \mathbb{D}$ is a bounded entire function, hence constant. Therefore either f is constant or its image misses at most one point. □

With Picard's theorem in hand, we can note that if $\Omega \subset \mathbb{C}$ is not all of $\mathbb{C}$, it is missing at least one point. (We will try to use Ω for an arbitrary domain for today in place of D, to avoid confusion with $\mathbb{D}$.) If $\Omega = \mathbb{C} \setminus \{c\}$ for some $c \in \mathbb{C}$, it is not simply connected, so we can assume Ω is missing at least two points. Then there is no non-constant analytic map $\mathbb{C} \to \Omega$, by Picard's theorem, so $\mathbb{C}$ and Ω cannot be conformally equivalent. Therefore by the Riemann mapping theorem, Ω is conformally equivalent to the unit disk $\mathbb{D}$.

Our goal today is to try to justify this concretely: for various different kinds of domains Ω, we would like to find conformal maps $\Omega \to \mathbb{D}$.

We have already seen that the upper half-plane $\mathcal{H}$ admits a conformal map $\mathcal{H} \to \mathbb{D}$, given by $z \mapsto \frac{z-i}{z+i}$. Thus $\mathcal{H}$ is conformally equivalent to $\mathbb{D}$, so to show that a domain Ω has a conformal map to $\mathbb{D}$, we can equally well show that it has a conformal map to $\mathcal{H}$. This will sometimes be useful.

We have been vague about the term "simply connected," but let's define it for our purposes as a domain Ω such that for any simple closed curve γ in Ω, γ can be continuously deformed to a point. This includes for example all star-shaped domains, but does not include annuli. Every domain listed below will be simply connected.

The first kind of domain we want to study is a generalization of the upper half-plane: a sector is a domain of the form $\Omega = \{z \neq 0 : \alpha < \arg z < \beta\}$ for fixed real $\alpha < \beta$. For example, taking $\alpha = 0$ and $\beta = \pi$ gives the upper half-plane.

By rotating by $-\alpha$, we can always assume $\alpha = 0$, so we may as well assume our sector is of the form $\{0 < \arg z < \alpha\}$ for some $\alpha < 2\pi$. The principle branch of $z \mapsto z^{A/\alpha}$ sends Ω

to $\{0 < \arg z < A\}$, so in particular $z \mapsto z^{\pi/\alpha}$ sends Ω to the upper half-plane. Away from $z = 0$, it is conformal and bijective, so this gives a conformal mapping $\Omega \to \mathcal{H}$, which we can compose with the standard map $\mathcal{H} \to \mathbb{D}$ to get a map $\Omega \to \mathbb{D}$. Therefore any sector is conformally equivalent to $\mathbb{D}$.

Our next type of domain is a strip, which can be geometrically thought of as the region between two parallel lines in $\mathbb{C}$. By rotating, we can change any strip to a horizontal strip, which is of the form $\{a < (z) < b\}$ for some fixed real numbers $a < b$.

The map $z \mapsto e^z$ then sends this strip to $\{e^a < \arg z < e^b\}$, a sector. Since we know we can map any sector conformally onto the unit disk and $z \mapsto e^z$ is conformal, this gives a conformal map $\Omega \to \mathbb{D}$ as desired.

Let's try to work out a concrete example. Let D be the vertical strip $\{-1 < \mathrm{Re}(z) < 1\}$. To turn it into a horizontal strip, we multiply by i, which sends D to $\{-1 < \mathrm{Im}(z) < 1\}$. (More explicitly, if $z = x + iy$ with $-1 < x < 1$, then $iz = -y + ix$ with imaginary part x satisfying the same bounds.) We can then exponentiate to get $e^{iz} \in \{-1 < \arg z < 1\}$, a sector. Rotating by an angle of 1, we get $e^{iz+i} \in \{0 < \arg z < 2\}$, and taking the power of $\pi/2$ we get $e^{\pi iz/2+\pi i/2} \in \{0 < \arg z < \pi\}$, the upper half-plane. Finally composing with the standard map we get

$$z \mapsto \frac{e^{\pi iz/2+\pi i/2} - i}{e^{\pi iz/2+\pi i/2} + i} = \frac{e^{\pi iz/2} - 1}{e^{\pi iz/2} + 1}.$$

This map sends $0 \mapsto 0$, and in the limits $+i\infty \mapsto -1$, and $-i\infty \mapsto 1$.

Note that in general these maps $D \to \mathbb{D}$ are not unique. Indeed, for any such map f, we can compose with a conformal self-map of $\mathbb{D}$ to get a new map $D \to \mathbb{D}$. However, if we fix some conformal map $f : D \to \mathbb{D}$ and $g : D \to \mathbb{D}$ is another such map, then $f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$ is a conformal self-map of $\mathbb{D}$, which we classified last time; such maps are parametrized by $a \in \mathbb{D}$ and $\lambda = e^{i\theta} \in \partial\mathbb{D}$, so the space of conformal maps $D \to \mathbb{D}$ is also parametrized by this data.

The final type of domain we want to consider today is a lunar domain. Although this is not something we see often, it is actually a natural generalization of a strip: we've seen before that a straight line in $\mathbb{C}$ can be thought of as a special kind of circle on $\mathbb{C} \cup \{\infty\}$, with the other circles corresponding to circles in $\mathbb{C}$, i.e. "a straight line is a circle with infinite radius." A strip was the area bounded between two parallel straight lines in $\mathbb{C}$; a lunar domain D is a domain bounded between two curves in $\mathbb{C}$ which is each either a straight line or the arc of a circle, which intersect at two (distinct) points z_0 and z_1.

If we were to allow z_0 and z_1 to go to infinity in opposite directions, we could think of the resulting lunar domain as a strip, with both arcs given by parallel straight lines. Since we don't need to include z_0 and z_1 in D, only in the bounds, this makes more sense than one might expect; formally we can justify it by showing that there is a conformal map between a lunar domain and a strip, which by our results above boils down to showing there is a conformal map from a lunar domain to $\mathbb{D}$.

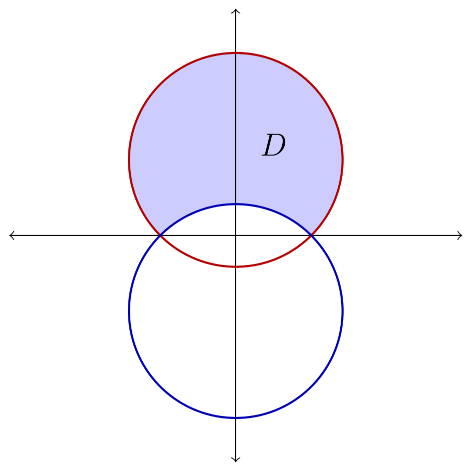
Another version of this manipulation is to take $z_0 = 0$ and $z_1 \to \infty$ (in some direction). In this case we have two rays meeting at 0, which we also think of as meeting at infinity; this gives a sector! Let's actually specify a conformal map from a lunar domain to a sector.

If $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a conformal map—i.e. a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ with nonvanishing derivative, exactly one pole, and whose image misses a single value c such that $f(\infty) = \lim_{z\to\infty} f(z) = c$, *or* an entire function $f : \mathbb{C} \to \mathbb{C}$ with $f(\infty) = \infty$ (a pole at infinity, i.e. in the case $c = \infty$), in both cases with nonvanishing derivative—sending $z_0 \mapsto 0$ and $z_1 \mapsto \infty$, then f will send D to this sector. Concretely, such a map f is given by

$$f(z) = \frac{z - z_0}{z - z_1},$$

a fractional linear transformation. Since we know sectors map conformally to $\mathbb{D}$, this shows that lunar domains do as well.

Let's give a concrete example of this. Let Γ_+ and Γ_- be circles centered at $\pm i$ respectively with radii $\sqrt{2}$.



One can compute that the intersection of the circles is at ± 1, so $z_0 = -1$, $z_1 = 1$, and so per the above we should first study the image of D under $f(z) = \frac{z+1}{z-1}$. Since both circles pass through ± 1 and f sends these points to 0 and ∞, f sends both circles to straight lines passing through ∞. Note that Γ_+ contains the point $i(1 + \sqrt{2})$, which is taken by f to $\frac{1}{\sqrt{2}}(1 - i) = e^{-\pi i/4}$, so this is the line through the origin of slope -1. Similarly, Γ_+ contains $-i(1 + \sqrt{2})$, which is taken by f to $e^{\pi i/4}$, so Γ_- is taken to the line of slope 1 through the origin.

Finally, $f(i) = -i$, so the image of D is the sector $\{\frac{5\pi}{4} < \arg z < \frac{7\pi}{4}\}$.