Lecture 24: the Schwarz lemma

Complex analysis, lecture 4
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We start off with the following relatively straightforward fact.

Proposition (Schwarz lemma)**.** *Let f be an analytic function on $D = \{z : |z| < 1\}$ with $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D$. Then $|f(z)| \leq |z|$ for all $z \in D$. Further, if there is some $0 \neq z \in D$ such that $|f(z)| = |z|$, then $f(z) = \lambda z$ for some constant λ with $|\lambda| = 1$.*

Proof. Since f is analytic with a zero at $z_0 = 0$, this is a zero of order at least 1, so we can write $f(z) = zg(z)$ for some analytic function g on D. It then suffices to show that $|g(z)| \leq 1$.

We use the maximum principle. Choose some $0 < r < 1$, and note that if $|z| = r$, then $|g(z)| = |f(z)/z| = |f(z)|/r \leq 1/r$ by the hypothesis on f. Therefore by the maximum principle, we have $|g(z)| \leq \frac{1}{r}$ for all $|z| \leq r$. Taking the limit as $r \to 1$, we obtain $|g(z)| \leq 1$ as desired, so $|f(z)| \leq |z|$.

Finally, if $|f(z)| = |z|$ for $0 \neq z \in D$, then $|g(z)| = 1$, so by the strict maximum principle g is constant, so f must be as above. □

By the same method we could prove an analogous statement on any disk: if f is analytic on $D = \{z : |z - z_0| < R\}$ with $f(z_0) = 0$ and $|f(z)| \leq M$, then

$$|f(z)| \leq \frac{M}{R}|z - z_0|,$$

and if equality holds at any point $z \neq z_0$ then $f(z) = \lambda(z - z_0)$ for some λ with $|\lambda| = \frac{M}{R}$. (We could also obtain this directly from the Schwarz lemma, by suitable changes of coordinates and scaling.)

We can think of the Schwarz lemma as quantifying the notion of continuity: a continuous function f is one such that as z gets closer to z_0, $f(z)$ also gets closer to $f(z_0)$. The Schwarz lemma tells us for an analytic function exactly how fast it has to get closer.

Under the same hypotheses as the Schwarz function, we can replace the conclusion with a statement about the derivative:

Proposition (Infinitesimal Schwarz lemma)**.** *Let f be an analytic function on $D = \{z : |z| < 1\}$ with $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D$. Then $|f'(0)| \leq 1$, and if $|f'(0)| = 1$ then $f(z) = \lambda z$ for some constant λ with $|\lambda| = 1$.*

Note that this is not automatic: one cannot differentiate inequalities! In this case however $|f(z)| \leq |z|$ implies $\left|\frac{f(z)}{z}\right| \leq 1$, and taking the limit on the left as $z \to 0$ gives $|f'(0)|$.

Note that this is actually a special case of Cauchy's estimates. In fact, it's actually weaker: here we had to assume that $f(0) = 0$, where we wouldn't need any assumption on $f(0)$ for Cauchy's estimates. However this version was much easier to prove.

Our main application of the Schwarz lemma will be to studying conformal self-maps of the unit disk. Let's quickly recall what we mean by this. We temporarily write $\mathbb{D}$ for the open unit disk $\{z : |z| < 1\}$. Recall that a conformal map $\mathbb{D} \to \mathbb{D}$ is a smooth bijection $\mathbb{D} \to \mathbb{D}$ (i.e. a smooth function $\mathbb{D} \to \mathbb{C}$ whose image is contained in $\mathbb{D}$) that preserves angles; for example, rotations, translations, and scalings are conformal, but something like $f(z) = z^2$ is not conformal at 0, since it doubles angles around it.

We saw the criterion that analytic bijections with nowhere vanishing derivative are conformal maps, and in fact later on that the converse holds as well. So we are interested in studying analytic bijections $\mathbb{D} \to \mathbb{D}$ with derivative nonzero on $\mathbb{D}$.

With this in mind, we can prove the following lemma.

Lemma. *Let $f : \mathbb{D} \to \mathbb{D}$ be a conformal map such that $f(0) = 0$. Then f is a rotation, i.e. $f(z) = e^{i\theta} \cdot z$ for some fixed $\theta \in [0, 2\pi)$.*

Proof. We start with the Schwarz lemma: since $f(0) = 0$ and $|f(z)| < 1$, it follows that $|f(z)| \leq |w|$. Since f^{-1} is also conformal and also satisfies $f^{-1}(0) = 0$ and $|f^{-1}(w)| < w$, we likewise have $|f^{-1}(w)| \leq |w|$. For $w = f(z)$, this reads $|z| \leq |f(z)|$, so combining this with the above we have $|f(z)| = |z|$, hence $f(z) = \lambda z$ for some $|\lambda| = 1$, i.e. $\lambda = e^{i\theta}$ as claimed. □

From here, it is not too big a step to completely classify the conformal self-maps of $\mathbb{D}$:

Theorem. *The conformal maps $\mathbb{D} \to \mathbb{D}$ are precisely the functions of the form*

$$f(z) = e^{i\theta} \frac{z - a}{1 - \overline{a}z}$$

for fixed $0 \leq \theta < 2\pi$, $a \in \mathbb{D}$.

Proof. First, we can note that these are in fact conformal maps: e.g. by our study of fractional linear transformations, or by directly computing the derivative. The hard part is showing that if f is a conformal self-map of $\mathbb{D}$, then it is of this form.

Let f be a conformal self-map of $\mathbb{D}$, $a = f^{-1}(0)$, and $g(z) = \frac{z-a}{1-\overline{a}z}$, so g is a conformal self-map of $\mathbb{D}$, with $g(a) = 0$. Then $f \circ g^{-1}$ is also a conformal self-map, with $f(g^{-1}(0)) = f(a) = 0$. Therefore by the lemma above, $f(g^{-1}(z)) = e^{i\theta} z$ for some θ. Letting $w = g^{-1}(z)$, so $z = g(w)$, this gives

$$f(w) = e^{i\theta} g(w) = e^{i\theta} \frac{w - a}{1 - \overline{a}w},$$

or changing variables back this gives the formula of the theorem. □

Note that θ and a are uniquely determined by f, so there is a bijection between conformal self-maps of $\mathbb{D}$ and pairs $(a, e^{i\theta}) \in \mathbb{D} \times \partial\mathbb{D}$.

As an application, we can prove a stronger form of Schwarz's lemma, called Pick's lemma.

Proposition (Pick's lemma)**.** *If f is analytic on $\mathbb{D}$ and $|f(z)| < 1$ for all $z \in \mathbb{D}$, then*

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}$$

for all $z \in \mathbb{D}$, and if equality holds at any point then f is a conformal self-map of $\mathbb{D}$, and equality holds at every point.

Note that if $f(0) = 0$, then this recovers $|f'(0)| \le 1$ as above; but it is much stronger in general, and now stronger than the corresponding Cauchy estimate. (For one thing, it gives a bound everywhere in the disk, not just at the central point.)

Proof. The idea is to compose with conformal self-maps of $\mathbb{D}$ to move both z and $f(z)$ to 0, so that we can apply the Schwarz lemma. More precisely, fix $z_0 \in \mathbb{D}$, and let $w_0 = f(z_0)$. We fix conformal maps

$$g(z) = \frac{z + z_0}{1 + \overline{z}_0 z}, \qquad h(w) = \frac{w - w_0}{1 - \overline{w}_0 w}$$

such that $g(0) = z_0$, $h(w_0) = 0$. Then $h \circ f \circ g$ sends 0 to 0, since $h(f(g(0))) = h(f(z_0)) = h(w_0) = 0$. We can therefore apply the infinitesimal Schwarz lemma to get $|(h \circ f \circ g)'(0)| = |h'(w_0)f'(z_0)g'(0)| \le 1$, or

$$|f'(z_0)| \le \frac{1}{|h'(w_0)g'(0)|} = \frac{1 - |w_0|^2}{1 - |z_0|^2}.$$

Substituting $z_0 = z$ and $w_0 = f(z)$ gives the desired bound.

If equality holds, this is the same thing as $|(h \circ f \circ g)'(0)| = 1$, which implies $h(f(g(z))) = \lambda z$ for $|\lambda| = 1$. This is in particular a conformal self-map of $\mathbb{D}$, so since h and g are as well, composing on the left and right with their inverses shows that f is also a conformal self-map of $\mathbb{D}$, and then by the formula for f from above we can check that the inequality is actually an equality for all z. □

Pick's lemma is important in hyperbolic geometry, where it leads to the notion of the hyperbolic length of a curve in $\mathbb{D}$. We briefly sketch this.

Naively, the length of a curve is given by $\int_\gamma |dz|$. However, under our principle that we should think of conformal maps as equivalences, this is not satisfying: it is not preserved under conformal maps, and if we're really thinking of these maps as equivalences then length should be the same on both sides.

Let $w = f(z)$ be a conformal self-map of $\mathbb{D}$. Then by Pick's lemma, $\left|\frac{dw}{dz}\right| = \frac{1-|w|^2}{1-|z|^2}$, or $\frac{|dw|}{1-|w|^2} = \frac{|dz|}{1-|z|^2}$. Thus a differential of this form is preserved under conformal maps; so if we define the hyperbolic length of γ to be

$$\int_\gamma \frac{|dz|}{1 - |z|^2},$$

this is genuinely invariant under conformal maps. (This is often rescaled by a factor of 2, which doesn't change this invariance.)

This leads to the notion of hyperbolic distance: for two points $z_0, z_1 \in \mathbb{D}$, the hyperbolic distance between them is the length of the shortest path connecting them. Here we should take the notion of hyperbolic length, so these paths (called geodesics) are not necessarily straight lines! (In fact, they are given by arcs passing through both points which intersect the unit circle at a 90° angle.) Then the hyperbolic distance, unlike the usual one, is preserved under conformal maps. Using this different notion of distance gives a new kind of geometry, which is hyperbolic geometry.

By taking a conformal mapping between the disk and some other space, one can get a new "model" for hyperbolic geometry; since we consider conformal maps to be an equivalence in this setting, this doesn't change any of the new geometry, but lets us think of it differently. We've already seen one such model: there is a conformal map between $\mathbb{D}$ and the upper half-plane $\mathcal{H}^+ = \{z : \mathrm{Im}\, z > 0\}$, which is another commonly used model.