Lecture 22: applications of the residue theorem

Complex analysis, lecture 4
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Today we continue with applications of the residue theorem to real integrals. Last time, we saw how to use it to compute integrals of rational functions over the real line, with the example of

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx = \pi.$$

To start with today, let's complicated this integral a little bit: what about something like

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx?$$

This is no longer amenable to calculus techniques: $\frac{\cos x}{x^2+1}$ does not have an elementary antiderivative. We can try the residue method from last time: let D be the upper half-disk of radius N, with boundary consisting of the interval $[-N, N]$ and the arc γ from N to $-N$, of radius N. Then

$$\int_{\partial D} \frac{\cos z}{z^2+1}\, dz = \int_{-N}^{N} \frac{\cos x}{x^2+1}\, dx + \int_{\gamma} \frac{\cos z}{z^2+1}\, dz.$$

We can evaluate the integral on the left via the residue theorem; the first integral on the right is the one we want to find, and then we can try to bound the second integral by the ML bound. Here however we encounter a problem: it is not clear that $\cos z$ should have a reasonable bound for $z \in \gamma$. Indeed, for example at $z = Ni$, which is on γ, we have

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{-N} + e^{N}),$$

which for N large becomes extremely large, much larger than the denominator of order N^2!

However, there is a trick we can use. For x real, $\cos x$ is the real part of e^{ix}. Therefore if we could compute the integral

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\, dx,$$

then we could determine our desired integral by taking the real part.

This is more amenable to our approach above because e^{iz} is actually nicely bounded on D: for $z = x + iy \in D \cup \partial D$, we must have $y \geq 0$, so $e^{iz} = e^{ix-y} = e^{-y}e^{ix}$ has absolute value $e^{-y} \leq 1$. Therefore, since for z on γ we have $|z| = N$ and so $\left|\frac{1}{z^2+1}\right| \leq \frac{1}{N^2}$ and γ has length πN, by the ML bound we find

$$\left|\int_{\gamma} \frac{e^{iz}}{z^2+1}\, dz\right| \leq \frac{\pi N \cdot 1}{N^2} = \frac{\pi}{N},$$

so this integral vanishes as $N \to \infty$. Therefore to compute our integral it suffices to evaluate

$$\int_{\partial D} \frac{e^{iz}}{z^2+1}\, dz.$$

Since e^{iz} is entire and nonzero, the only singularity of $\frac{e^{iz}}{z^2+1}$ in D is at $z = i$, where it has residue $\left.\frac{e^{iz}}{2z}\right|_{z=i} = \frac{1}{2ei}$. Therefore by the residue theorem the integral is

$$\frac{2\pi i}{2ei} = \frac{\pi}{e}.$$

Since this is real, taking the real part gives $\frac{\pi}{e}$ again and so we conclude that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx = \frac{\pi}{e}.$$

A different kind of example is for the integration of trigonometric functions. If θ ranges from 0 to 2π (or $-\pi$ to π, or any other interval of length 2π, then the usual mapping $\theta \mapsto z = e^{i\theta}$ lets us translated between integrals over θ and integrals around the unit circle. We have often used this method to turn complex line integrals into usual one-variable integrals over θ. Now though we have very powerful tools to evaluate complex line integrals, and we would rather go the other way to use these tools to evaluate real integrals.

The reason this is especially useful for trigonometric functions is that we can naturally write $\cos\theta$ and $\sin\theta$ in terms of z: we have $z = e^{i\theta} = \cos\theta + i\sin\theta$, and since $|z| = 1$ we also have $\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta = \overline{z}$, so

$$\cos\theta = \mathrm{Re}(z) = \frac{1}{2}(z + 1/z), \qquad \sin\theta = \mathrm{Im}(z) = \frac{1}{2i}(z - 1/z).$$

Let's work out an example. Fix a real number $a > 1$, and consider

$$\int_0^{2\pi} \frac{1}{a + \cos\theta}\, d\theta.$$

Writing $z = e^{i\theta}$, we have $dz = ie^{i\theta}\, d\theta = iz\, d\theta$, so $d\theta = \frac{1}{iz}\, dz$. Combining this with the above, we see that this integral is the same thing as

$$\int_{|z|=1} \frac{1}{a + \frac{1}{2}(z+1/z)} \cdot \frac{1}{iz}\, dz = -2i \int_{|z|=1} \frac{1}{1 + 2az + z^2}\, dz.$$

The integrand is a rational function, with poles at the zeros of $1 + 2az + z^2$; by the quadratic formula, these are at $z = -a \pm \sqrt{a^2-1}$. Since $a > 1$, we have $\sqrt{a^2-1}$ a positive real number, and $-a - \sqrt{a^2-1} < -1$ so it is not contained in the unit circle; however $a - 1 < \sqrt{a^2-1} < a$ since $a > 1$, so $-1 < -a + \sqrt{a^2-1} < 0$ and so $-a + \sqrt{a^2-1}$ is contained in the unit circle, and the integrand has a simple pole there, with residue

$$\left.\frac{1}{2z+2a}\right|_{z=-a+\sqrt{a^2-1}} = \frac{1}{2\sqrt{a^2-1}}.$$

Therefore by the residue theorem this is

$$-2i \cdot 2\pi i \cdot \frac{1}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}.$$

We could consider both sides of the equation

$$\int_0^{2\pi} \frac{1}{a+\cos\theta}\,d\theta = \frac{2\pi}{\sqrt{a^2-1}}$$

as a function of the variable a. Above, we have shown that the two sides are equal for $a \in (1,\infty)$. If we wanted to take a a complex number, we need to specify a branch of the square root on the right: we take the principal branch. Recall that the function $\sqrt{z^2-1}$ can be defined on $\mathbb{C} \setminus [-1,1]$, taking the branch cut from -1 to 1 along the real axis; away from this branch cut, it is continuous, and in fact analytic. This might lead us to guess that the equation above holds for $a \in \mathbb{C} \setminus [-1,1]$. (If $a \in [-1,1]$, then both sides are ill-defined: on the right because of the branch cut, and on the left because then there exists $\theta \in [0,2\pi]$ with $a + \cos\theta = 0$.)

Indeed, both sides define an analytic function of a on $\mathbb{C} \setminus [-1,1]$, after fixing the branch on the right as above; and we showed above that these functions agree on $(1,\infty)$, a subset containing non-isolated points. Hence they must be equal everywhere on $\mathbb{C} \setminus [-1,1]$.

This brings us to the more general question of applying the residue theorem to functions with branch cuts. As above, one can do this, but we need to be careful not to cross the branch cuts, and to keep track of our branches.

Let $-1 < a < 1$ be a real number, and consider the function

$$f(z) = \frac{z^a}{(1+z)^2}.$$

For nonnegative real numbers z, there is a unique branch of f with positive real values, so we can define the integral

$$\int_0^\infty \frac{x^a}{(1+x)^2}\,dx.$$

To use the residue theorem, though, we will need to specify a branch of f. We take the branch cut along the positive real line, and define f on $\mathbb{C} \setminus [0,\infty)$ by

$$f(re^{i\theta}) = \frac{r^a e^{ia\theta}}{(1+re^{i\theta})^2}$$

for $0 < \theta < 2\pi$. Note that this has phase factor $e^{2\pi i a}$, as can be seen by comparing the value of f at $\theta = 0$ and at $\theta = 2\pi$.

Fix real numbers $0 < r < R$, where we will eventually want to take $R \to \infty$ and $r \to 0$. Let D be the slit annulus $D = \{z \in \mathbb{C} \setminus [0,\infty) : r < |z| < R\}$. We think of the slit in $\mathbb{C} \setminus [0,\infty)$ as having a top edge, at $\theta = 0$, and a bottom edge, at $\theta = 2\pi$; so the boundary of D consists of a line from r to R along $\theta = 0$, on the top edge of the slit, the circle of radius

R in the positive direction, the line from R to r along the bottom edge $\theta = 2\pi$, and the circle of radius r in the negative direction. (This is sometimes called the keyhole contour.) We call these pieces respectively ℓ_+, γ_R, ℓ_-, and γ_r. In particular, when we take $r \to 0$ and $R \to \infty$, the limit of $\int_{\ell_+} f(z)\,dz$ will recover our desired real integral.

First, let's compute the integral

$$\int_{\partial D} f(z)\,dz$$

using the residue theorem. The only singularity of f in D is at $z = -1$, where it has a double pole with residue

$$\frac{d}{dz}(z+1)^2 f(z)\Big|_{z=-1} = \frac{d}{dz} z^a\Big|_{z=-1} = a(-1)^{a-1} = -ae^{\pi a i}.$$

Therefore by the residue theorem this integral is

$$-2\pi i a e^{\pi i a}.$$

Next, we'll study the components

$$\int_{\partial D} f(z)\,dz = \int_{\ell_+} f(z)\,dz + \int_{\gamma_R} f(z)\,dz + \int_{\ell_-} f(z)\,dz + \int_{\gamma_r} f(z)\,dz.$$

For the circular integrals, we use the ML bound: on γ_R, we have $L = 2\pi R$ and $|f(z)| \le \frac{R^a}{(R-1)^2}$, so $ML = 2\pi R^{a+1}/(R-1)^2 \sim 2\pi R^{a-1}$, which tends to 0 as $R \to \infty$ since $a < 1$, and on γ_r we have $L = 2\pi r$ and $|f(z)| \le \frac{r^a}{(r-1)^2} \sim r^a$, so $ML \sim 2\pi r^{a+1} \to 0$ as $r \to 0$ since $a > -1$. Therefore both circular integrals vanish in the limit.

For the line integrals, we know that f on ℓ_- is $e^{2\pi i a}$ times the value of f on ℓ_+, since this is the definition of the phase factor. In addition, ℓ_- is oriented in the opposite direction to ℓ_+. Therefore

$$\int_{\ell_+} f(z)\,dz + \int_{\ell_-} f(z)\,dz = (1 - e^{2\pi i a}) \int_{\ell_+} f(z)\,dz,$$

which in the limit is then equal to $-2\pi i a e^{\pi i a}$ as above. Therefore for $a \neq 0$

$$\lim_{R\to\infty} \lim_{r\to 0} \int_{\ell_+} f(z)\,dz = \int_{-\infty}^{\infty} \frac{x^a}{(1+x)^2}\,dx = -\frac{2\pi i a e^{\pi i a}}{1 - e^{2\pi i a}} = -\frac{2\pi i a}{e^{-\pi i a} - e^{\pi i a}} = \frac{\pi a}{\sin(\pi a)}.$$

(To cover the case $a = 0$ as well, the left-hand side can be seen by direct integration to be 1, and the limit of the right-hand side as $a \to 0$ is also 1, so we can say that this holds at $a = 0$ as well after removing a removable singularity.)

We can again extend this to complex values of a as above, but now we have obstructions at $a = \pm 1$, where both sides now fail to be analytic (the left-hand side diverges and the right-hand side has a zero in the denominator). On the strip $-1 < \operatorname{Re}(a) < 1$ however both sides are analytic, so since they agree on $(-1, 1)$, which contains non-isolated points, it follows that they agree everywhere on the strip.

The right-hand side extends to $a \in \mathbb{C} \setminus \{\ldots, -3, -2, -1, 1, 2, 3, \ldots\}$, but the left-hand side does not converge outside this strip. Therefore we could view the right-hand side as providing an analytic continuation of this integral to a meromorphic function on $\mathbb{C}$.