Lecture 20: isolated singularities

Complex analysis, lecture 4
October 31, 2025

1. Isolated singularities

In the past, although we've used the term "singularity" freely, we've been a little vague about what we mean by it. We could take the following, rather naive definition: a singularity of a function f is any point in $\mathbb{C}$ at which f is not analytic.

This works fine for all purposes so far, but is not totally satisfactory. For example, it could be that f is defined on some domain $D \subset \mathbb{C}$, e.g. some disk of finite radius, and outside of D it simply does not make sense to talk about f. In this case, per the definition above we would say that f has a singularity at every point in $\mathbb{C} \setminus D$, which does not match our intuition very well.

Instead, we typically want to think about functions which are *almost* analytic on $\mathbb{C}$ (or on some subdomain), away from some set of points. This set of points doesn't need to be finite, but it shouldn't be able to form dense regions: in particular every point should be isolated. (We sometimes in this case say that this set is itself isolated, or that it is discrete.)

This brings us to today's topic: we say that a point $z_0 \in \mathbb{C}$ is an isolated singularity of f if it is a singularity, i.e. f is not analytic at z_0, and there is some $r > 0$ such that f is analytic on the punctured disk $0 < |z - z_0| < r$. At the end of yesterday's class, we noted that every singularity of a meromorphic function (i.e. the ratio of two analytic functions) is isolated, so this is a useful notion.

Since f is defined on an annulus around z_0 as above (with radii 0 and r), we can study its Laurent series at z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Recall that this series is unique, so all of the data of f on this punctured disk is classified by the Laurent series. We can therefore study the singularity by studying the series.

We make the following definition: say that the isolated singularity of f at z_0 is

- *removable* if $a_n = 0$ for $n < 0$;
- a *pole* if there are only finitely many $n < 0$ such that $a_n \neq 0$;
- an *essential singularity* if there are infinitely many $n < 0$ such that $a_n \neq 0$.

So every isolated singularity is either removable, a pole, or an essential singularity. We study each of these types in turn.

Removable singularities

If $a_n = 0$ for $n < 0$, then the Laurent series becomes

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

and so the right-hand side is analytic at $z = z_0$, with value a_0. Thus although f a priori need not be analytic—or even defined—at z_0, we can find an analytic function $\widetilde{f}$ which is analytic at z_0 and such that for z in the punctured disk around z_0, $\widetilde{f}(z) = f(z)$, so this gives an analytic extension of f to z_0. Thus for example a removable singularity would not provide an obstruction to the radius of convergence of a Taylor series at a nearby point.

For example, consider

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{1}{6}z^3 + \cdots\right) = 1 - \frac{1}{6}z^2 + \cdots .$$

Although a priori f is not defined at 0, the Laurent series has no negative terms, so it extends to an analytic function at $z = 0$, and indeed on the whole complex plane.

If f has a removable singularity at z_0, then it is bounded near z_0, as can easily be seen from its Taylor expansion. Conversely, one can show that if it is bounded near z_0, then in order to prevent the $(z - z_0)^n$ terms from exploding to infinity for $n < 0$, we must have $a_n = 0$ for all $n < 0$, i.e. f has a removable singularity at z_0.

Poles

If there are finitely many $n < 0$ such that $a_n \neq 0$, we can choose the largest integer N such that $a_{-N} \neq 0$, i.e. for all $n > N$ we have $a_{-n} = 0$. In this case

$$\begin{aligned} f(z) &= \frac{a_{-N}}{(z - z_0)^N} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots \\ &= \frac{1}{(z - z_0)^N} \left(a_{-N} + a_{-N+1}(z - z_0) + \cdots\right), \end{aligned}$$

i.e. there exists a function

$$h(z) = a_{-N} + a_{-N} + 1(z - z_0) + \cdots$$

which is analytic at z_0 with $h(z_0) = a_{-N} \neq 0$ such that

$$f(z) = \frac{h(z)}{(z - z_0)^N}.$$

In this case we say that f has a pole of order N at z_0.

Note the similarity of this definition to the definition of a zero of order N. Indeed, we claim that $f(z)$ having a pole of order N at z_0 is equivalent to $\frac{1}{f(z)}$ having a zero of order N at z_0: indeed, if $f(z) = \frac{h(z)}{(z-z_0)^N}$ with $h(z_0) \neq 0$ and h analytic at z_0, then

$$\frac{1}{f(z)} = (z - z_0)^N \frac{1}{h(z)},$$

and $\frac{1}{h}$ is analytic and nonzero at z_0 since h is, so $\frac{1}{f}$ has a zero of order N at z_0. The converse is similar.

Like for zeros, we refer to a pole of order 1 as a simple pole, and to a pole of order 2 as a double pole and so on. So for example $f(z) = \frac{1}{z}$ has a simple pole at $z = 0$, with Laurent expansion simply

$$f(z) = z^{-1},$$

with only one term. A slightly less trivial example is $f(z) = \frac{1+z}{z^2}$, with Laurent expansion

$$f(z) = \frac{1}{z^2} + \frac{1}{z},$$

which has a double pole at $z = 0$.

Similarly to the case of f having a zero of order 0, we sometimes say that f has a zero of order 0 at z_0 if the above holds with $N = 0$, i.e. if f is analytic (and nonzero) at z_0. Note this is actually the same thing as having a zero of order 0. We can further unify the definitions of poles and zeros of given order: a pole of order N is equivalent to a zero of order $-N$, and vice versa.

It is sometimes useful to be able to refer to the non-analytic portion of a function f having a pole at z_0: let

$$P(z) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n$$

be the principal part of f at z_0. Then $f(z) - P(z)$ is analytic at z_0. This $P(z)$ is the f_1 in our decomposition from last time. When f has a pole at z_0, P has only finitely many terms, so it is a polynomial in $(z - z_0)^{-1}$.

We showed that every zero of an analytic function (which is not identically zero) is isolated with finite order. As a corollary, we conclude that every singularity of a meromorphic function is a pole, since locally we can write it as an analytic function divided by some power of $z - z_0$. In fact, one can also show the converse: if a function is analytic on $\mathbb{C}$ away from a discrete set of points, at each of which it has a pole, then it is meromorphic. Indeed, it suffices to show that it is locally in a punctured disk around each point given by a ratio of holomorphic functions; and at a pole of order N, it can be written as

$$f(z) = \frac{h(z)}{(z - z_0)^N}$$

as above, which is meromorphic.

Essential singularities

The "worst" type of singularity is an essential singularity. We saw above that these do not arise from meromorphic functions, but they can arise from natural compositions of them: a common example is

$$f(z) = e^{1/z}$$

at $z = 0$, with Laurent series given by

$$f(z) = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots,$$

with infinitely many negative powers of z. Although these singularities are not poles, we can often treat them as "poles of infinite order," and will sometimes do so going forward.

2. Isolated singularities at ∞

We can also formulate this notion at the point at infinity, just like we did for zeros at infinity. If $g(z) = f(1/z)$, then we say that f has an isolated singularity at infinity if g has an isolated singularity at 0; or equivalently, if f is analytic on $\{|z| > R\}$ for some R, but is not (necessarily) analytic at infinity.

Letting

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^{-n}$$

be the Laurent expansion at infinity, we again say that the singularity at infinity is removable if $a_n = 0$ for $n < 0$; a pole if $a_n = 0$ for all but finitely many $n < 0$; and otherwise an essential singularity. A function with a removable singularity at infinity is analytic at infinity; $f(\infty)$ is already defined by a limit, so it's rarely useful to distinguish between the two notions. Poles an essential singularities at infinity however are common.

For example, any rational function $\frac{P(z)}{Q(z)}$ for polynomials P, Q is either analytic at infinity (if $\deg P \leq \deg Q$) or has a pole (with order $\deg Q - \deg P$, if this is positive). On the other hand, a function like $f(z) = e^z$ has an essential singularity at ∞, just as $e^{1/z}$ had an essential singularity at 0.

One can generalize these examples to show that if f is a meromorphic function on $\mathbb{C}$—i.e. analytic away from a discrete set, possibly empty, at which it has poles—and it is analytic or has a pole at infinity (in other words, if f is meromorphic on $\mathbb{C} \cup \{\infty\}$), then it is actually a rational function, so a non-rational meromorphic function on $\mathbb{C}$ must have an essential singularity at ∞.

We quickly sketch the argument, which involves a little bit of topology: since $\mathbb{C} \cup \{\infty\}$ is compact, any infinite set of points inside of it has some limit point, either finite or infinite, so if the set of poles of f was infinite then it could not be discrete. Hence f must have finitely many poles $z_1, \ldots, z_m$ in $\mathbb{C}$, together with possibly a pole at infinity.

Multiplying by suitable factors of $(z - z_i)^{N_i}$ to get rid of each pole in $\mathbb{C}$, and by z^{-N_∞} to get rid of the possible pole at infinity, we obtain a function $g(z)$ which is analytic everywhere on $\mathbb{C}$ and analytic at infinity, hence bounded. Therefore by Liouville's theorem g is constant. On the other hand, we obtained g from f by multiplying by finitely many rational functions, so since g is a constant f must be rational.

By going through this more carefully, one can find a formula for f as the sum of the principal parts at each pole (each of which is a rational function). This process can be used to find the partial fractions decomposition of any rational function, which came up last class, and which should also be familiar from calculus.