Lecture 2: stereographic projection and first branch cuts

Complex analysis, lecture 4
August 29, 2025

1. Stereographic projection

It is often helpful to "complete" the complex plane to get the "extended complex plane" by adding a point at infinity, i.e. $\mathbb{C} \cup \{\infty\}$. How should we think of this space?

We claim we should think of it as a sphere, "folding up" the plane and joining all the "edges" (infinitely far from the origin) to a single point ∞. We'll make sense of this by writing down an explicit bijection between $\mathbb{C} \cup \{\infty\}$ and the unit sphere $S^2 = \{X, Y, Z \in \mathbb{R} : X^2 + Y^2 + Z^2 = 1\}$.

The idea is this. Let $N = (0, 0, 1) \in S^2$ be the "north pole" of the sphere, and $P = (X, Y, Z)$ any other fixed point. We can draw a line passing through these two points; since the only point on the sphere with Z-coordinate equal to 1 is N, for $P \neq N$ this line is not parallel with the X-Y-plane (i.e. the plane $Z = 0$) and so intersects it at some point z. We claim that the map $P \mapsto z$ is a bijection from $S^2 \setminus \{N\}$ to $\mathbb{C}$; so we can think of ∞ as corresponding to N, giving a bijection $S^2 \to \mathbb{C} \cup \{\infty\}$. (Note that this is really about the plane $\mathbb{R}^2$, and doesn't a priori involve the complex structure!)

Before proving this, let's think about some examples to get a feel for this mapping. If we took P to be the south pole, $(0, 0, -1)$, then the line through N and P passes through the X-Y-plane at $z = (0, 0)$. If P is itself on this plane, then $z = P$ (or rather, if $P = (X, Y, 0)$, then $z = (X, Y)$). As P approaches N, the corresponding point z tends to infinity, at least in absolute value, so it makes some sense to say N should correspond to the point at infinity.

How should we go about proving this? Let's start by finding a formula for z. Points on the line connecting N and P are given by $(1 - t) \cdot N + t \cdot P = (tX, tY, 1 - t + tZ)$; we're looking for the point on this line with last coordinate zero, i.e. $1 - t + tZ = 1 - t(1 - Z) = 0$, so $t = \frac{1}{1-Z}$. Therefore z is given by the first two coordinates: $z = (X/(1 - Z), Y/(1 - Z))$. In the complex language,

$$z = \frac{X}{1 - Z} + \frac{Y}{1 - Z} i.$$

We can now construct an inverse, showing that this is in fact a bijection. Given $z = (x, y)$, we want to find (X, Y, Z) satisfying $X^2 + Y^2 + Z^2 = 1$ such that $x = X/(1 - Z)$ and $y = Y/(1 - Z)$. Recalling the notation $t = \frac{1}{1-Z}$, we multiply the defining equation for the sphere by t^2 with $x = tX$, $y = tY$, and $tZ = \frac{Z}{1-Z} = t - 1$, we get

$$x^2 + y^2 + (t - 1)^2 = t^2,$$

and solving for t gives

$$t = \frac{1}{2}(x^2 + y^2 + 1) = \frac{1}{2}(|z|^2 + 1).$$

Using the equations above, we solve to get

$$X = x/t = \frac{2x}{|z|^2 + 1},$$

$$Y = y/t = \frac{2y}{|z|^2 + 1},$$

$$Z = \frac{t-1}{t} = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

So we can invert the transformation.

In addition to giving us a new way to think of the (extended) plane, this correspondence also satisfies some nice properties. For example, longitudinal lines on the sphere map to straight lines on the plane, while latitudinal lines map to circles in the plane. More generally, every circle on the sphere maps to either a circle or a line in the plane, and vice versa lines and circles on the plane correspond to circles on the sphere; when we add the point at infinity, we can think of straight lines in the plane as circles passing through the point at infinity. The extended complex numbers, thought of as a sphere, are often called the Riemann sphere.

2. Squares and square roots

When dealing with real functions, there's a standard way to visualize them, namely by graphing them: we put the domain on one axis and the codomain on the other, and plot the graph of the function on the resulting two-dimensional space. When we're working with complex functions, both the domain and codomain are given by the complex plane, so the resulting space would be four-dimensional, which is much harder to visualize; so we need another approach. There are many ways of doing this, such as using color or graphing the real and imaginary parts separately. We want to introduce another: graphing how the values of the function change as the inputs change.

Let's work with the function $f(z) = z^2$. This is easiest to think about in polar coordinates: if $z = re^{i\theta}$, then $f(z) = z^2 = r^2 e^{2i\theta}$, so f squares the modulus and doubles the argument.

Graphically, squaring the modulus is straightforward enough in terms of scaling, so let's think about points on the unit circle. Here as z moves around the unit circle, i.e. as θ goes from $-\pi$ to π, z^2 moves around the unit circle twice as fast: the argument goes from -2π (equivalently, zero) to 2π (again) in this same period. So if we just wanted z^2 to go around the unit circle, from $-\pi$ to π, we should take θ from $-\pi/2$ to $\pi/2$.

In particular, if we wanted to find an inverse for f, we would need make a restriction something like this. That is: writing $z = re^{i\theta}$ with $-\pi < \theta \leq \pi$, we can find a square root of z given by $z = \sqrt{r} e^{i\theta/2}$.

We note though that something weird happens near the ray $\theta = \pi$: if $\theta = \pi - \epsilon$ for some very small positive ϵ, then $e^{i\theta}$ is very close to $e^{\pi i} = -1$, and in this model its square root $e^{i\theta/2}$ is very close to $e^{\pi i/2} = i$. However, if we instead looked at $e^{-i\theta} = e^{i(\epsilon - \pi)}$, which is also very close to -1 but on the other side of the real axis, we would find that its square root is $e^{-i\theta/2}$ which is very close to $e^{-\pi i/2} = -i$. So this is quite far from $\sqrt{e^{i\theta}} = e^{i\theta/2}$, even as $\epsilon \to 0$. In other words, this square root is not continuous!

Now, it only fails to be continuous near this ray $\theta = \pi$; elsewhere everything is fine. You might point out that this failure of continuity is only due to our particular choice of

$-\pi < \theta \leq \pi$, we could have chosen a different parametrization; but that would just move the discontinuity somewhere else.

To avoid having to think of this as a discontinuity, we make a "branch cut" in the complex plane along the negative real axis. If we think of this as a boundary, so we no longer think of points like $1 + i\epsilon$ and $1 - i\epsilon$ as close to each other, then we can now describe our square root function as a continuous function on this slit plane.

Now, observe that our square root $z = re^{i\theta} \mapsto \sqrt{r}e^{i\theta/2}$ has image in complex numbers with argument between $-\pi/2$ and $\pi/2$. Translating back to Cartesian coordinates, this is equivalent to having positive (or at least nonnegative[1]) real part. Restricted to nonnegative real numbers, this gives the positive square root. But we could also take the negative square root, which in the complex setting has image in complex numbers with argument less than or equal to $-\pi/2$ or greater than $\pi/2$. To make this nicer, we could translate by 2π and say these have argument between $\pi/2$ and $3\pi/2$.

These options for the square root function, which is a priori multivalued, are called its branches; let's call the "positive" branch we wrote down above f_1^{-1}, given by $f_1^{-1}(re^{i\theta}) = \sqrt{r}e^{i\theta/2}$, and the other branch f_2^{-1}, which is given by $-f_1^{-1}$. Each of these naturally lives on the slit complex plane we described above. If we wanted to be able to cross the negative real axis, we would have to combine these two branches somehow.

Let's return to the example above, $z = e^{i\theta}$ where $\theta = \pi - \epsilon$. Here $f_1^{-1}(z)$ was near i. As $\epsilon \to 0$, so $z \to -1$ from the positive imaginary side, this approaches i; but when ϵ becomes negative, so z crosses the negative real axis, then $f_1^{-1}(z)$ jumps to near $-i$, as we saw before. However, $f_2^{-1}(z) = -f_1^{-1}(z)$ is then near $+i$, i.e. near the value of z on the other side! So if we wanted to define a single inverse to f, we would take our two copies of the slit plane and glue them together: the top edge of the slit on the first plane is glued to the bottom edge on the second plane, and vice versa.

This space is a little hard to imagine; by thinking about how you deform this space, you can shape it into a sphere with two punctures. We can see this more concretely as follows: the function $f(z) = z^2$ gives a bijection between this space and $\mathbb{C} \setminus \{0\}$ (we have to exclude $z = 0$ to make sure the polar representation is well-defined, and indeed at zero the square root is only single-valued), and by stereographic projection we can think of $\mathbb{C} \setminus \{0\}$ as $(\mathbb{C} \cup \{\infty\}) \setminus \{0, \infty\} \simeq S^2 \setminus \{0, \infty\}$.

This is the Riemann surface for the square root function. We'll see more examples of Riemann surfaces next week; note that they don't appear on homework 1, because some of the relevant material is scheduled for the end of next week and so you would have very little time to do those problems, but they will appear on the following homework.

[1] There is some subtlety when the real part is exactly zero: then we include the positive imaginary axis but not the negative one, i.e. xi for $x \geq 0$.

3. The complex exponential

Finally, let's return to our definition of the exponential function $z \mapsto e^z$ in the complex setting. If $z = x + iy$, then we set

$$e^z = e^x \cdot e^{iy} = e^x \cdot (\cos y + i \sin y)$$

where e^x is as usual for a real number x. We saw when studying polar representations that this satisfies the additivity property $e^{a+b} = e^a \cdot e^b$: on the real component, this is a standard property, and on the imaginary component this means that rotating by y_1 and then by y_2 is the same as rotating by $y_1 + y_2$. We can likewise confirm properties such as

$$e^{-z} = \frac{1}{e^z}, \qquad (e^a)^b = e^{ab}.$$

We see that the exponential function is easiest to define in terms of Cartesian coordinates. On the other hand, its output is easiest to understand in terms of polar coordinates: while computing its real and imaginary parts involves the use of trigonometric functions, we can quickly see that $|e^{x+iy}| = e^x$ while $\arg(e^{x+iy}) = y$. So when we repeat our idea of comparing how changes in the domain translate to changes in the codomain, we use changes in Cartesian coordinates, i.e. in the real and imaginary part, on the domain, and changes in polar coordinates on the codomain. Changes in the real part scale the modulus by an exponential factor, so horizontal lines in the complex plane map to rays in the codomain; and changes in the imaginary part shift the argument, so vertical lines map to circles.

Unlike the square root function, the exponential function is not multivalued, so we don't need to worry about its Riemann surface. However, it is *not* injective, unlike in the real setting: for example, $e^{2\pi i} = e^0 = 1$. More generally, $f(z) = e^z$ is *periodic* with period $2\pi i$: that is, $f(z) = f(z + 2\pi i)$ for all z. This also means that $f(z + 2\pi i n) = f(z)$ for every integer n, so the preimage of $f(z)$ contains infinitely many points; so its inverse function will be (very) multivalued! We'll worry about this more next time.