Lecture 19: Laurent expansions

Complex analysis, lecture 4

October 29, 2025

The basic idea of Taylor expansions is to write a given analytic function as a power series, i.e. a linear combinations of z^n for nonnegative n. If we instead allowed n to be any integer, positive or negative, then what we would get is called a Laurent series expansion. Our main goal for today is to introduce this notion, and see how we can compute it.

When working with power series, the main geometric object is a disk centered at our point of interest z_0, with some radius (possibly 0 or $+\infty$). For Laurent series, the main object will be an *annulus*, which can be thought of as a ring, or a disk with both an inner and outer radius: if we fix $0 \le r < R \le +\infty$, the corresponding annulus centered at z_0 is the region $\{z \in \mathbb{C} : r < |z - z_0| < R\}$. In the special case where $r = 0$, this is $\{0 < |z - z_0| < R\} = \{|z - z_0| < R\} \setminus \{z_0\}$, and is sometimes called the punctured disk of radius R around z_0.

Note that unlike for disks, any annulus centered at z_0 does not contain the point z_0. This is important: often our functions of interest will not be analytic at z_0.

Fixing $0 \le r < R \le +\infty$ and an annulus D of radii r, R centered at z_0, suppose that $f : D \to \mathbb{C}$ is an analytic function. Let $D_0 = \{z : |z - z_0| < R\}$ and $D_1 = \{z : |z - z_0| > r\}$, so that $D = D_0 \cap D_1$.

Proposition. *With notation as above, there exist unique analytic functions $f_0 : D_0 \to \mathbb{C}$, $f_1 : D_1 \to \mathbb{C}$ with f_1 analytic at infinity with $f_1(\infty) = 0$ such that for $z \in D$,*

$$f(z) = f_0(z) + f_1(z).$$

The condition that $f_1(\infty) = 0$ isn't strictly necessary, one could also find a decomposition with a different value of $f_1(\infty)$; but imposing this condition makes the decomposition unique. (Otherwise, we could just subtract a constant from f_0 and add it to f_1 to get a different decomposition.)

Note that if f were analytic on all of D_0, not just on D, then we could simply take $f_0 = f$ and $f_1 = 0$. Similarly, if f were analytic on D_1 and at ∞, we could take $f_1 = f - f(\infty)$ and $f_0 = f(\infty)$.

Proof. First, we show that if such functions f_0, f_1 exist, then they are unique. Suppose that there was another pair $g_0 : D_0 \to \mathbb{C}$, $g_1 : D_1 \to \mathbb{C}$ of analytic functions with g_1 analytic at infinity and $f(z) = f_0(z) + f_1(z) = g_0(z) + g_1(z)$ for $z \in D$.

Define a function $h : \mathbb{C} \to \mathbb{C}$ by $h(z) = f_0(z) - g_0(z)$ for $z \in D_0$ and $h(z) = g_1(z) - f_1(z)$ for $z \in D_1$; note that by the above equation, if $z \in D = D_0 \cap D_1$ then the two definitions agree, so this is well-defined. Since the f_i and g_i are analytic where they are defined, h is analytic everywhere. As $z \to \infty$, since f_1 and g_1 are analytic at infinity with value 0, $h(\infty) = \lim_{z\to\infty} h(z) = 0$, so h is bounded, hence by Liouville's theorem h is constant, and since $h(\infty) = 0$ we must have $h(z) = 0$ for all z. Hence $f_0(z) = g_0(z)$ and $g_1(z) = f_1(z)$, i.e. f_0 and f_1 are unique.

Now we show that such f_i actually exist. Choose some $r < r' < R' < R$. For $r' < |z - z_0| < R'$, we can take the sub-annulus of radii r', R' centered at z_0, which still contains z, and its boundary is the union of the circles of radii r' and R', on which f is still analytic. (We need to choose r' and R' rather than just using r and R to make sure that f extends to the boundary.) Therefore by Cauchy's formula

$$f(z) = \frac{1}{2\pi i} \int_{|z-z_0|=R'} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{|z-z_0|=r'} \frac{f(w)}{w-z}\, dw.$$

Let

$$f_0(z) = \frac{1}{2\pi i} \int_{|z-z_0|=R'} \frac{f(w)}{w-z}\, dw, \qquad f_1(z) = -\frac{1}{2\pi i} \int_{|z-z_0|=r'} \frac{f(w)}{w-z}\, dw.$$

Then for $r' < |z - z_0| < R'$ we have $f(z) = f_0(z) + f_1(z)$; f_0 is analytic on $|z - z_0| < R'$, and f_1 is analytic on $|z - z_0| > r'$ with the path viewed as the boundary of $\{z : |z - z_0| > r'\}$, and is analytic at infinity with value 0 since $\lim_{z\to\infty} f_1(z) = 0$. (One could also make a more formal argument by evaluating on $1/z$ and taking a limit.) We are still using r' and R' instead of r and R, but could conclude by taking the limit as $r' \to r$ and $R' \to R$, or by using the uniqueness part above to observe that the result is independent of r' and R'. $\square$

Before proceeding let's work out an example, which demonstrates the importance of fixing the annulus. Consider $f(z) = \frac{1}{(z-1)(z-2)}$ and $z_0 = 0$. There are at least three different annuli we could consider on which f is analytic: $0 < |z| < 1$; $1 < |z| < 2$; and $2 < |z|$.

On the first annulus, which is a punctured disk, note that f actually extends to an analytic function on $|z| < 1$, so we can take $f_0(z) = f(z)$ and $f_1(z) = 0$.

Similarly on the third annulus, f is analytic at infinity with value $\lim_{z\to\infty} f(z) = 0$, so we can just take $f_0(z) = 0$ and $f_1(z) = f$.

The most interesting case is the second annulus, where we cannot extend to $|z| < 2$ since we have a pole at $z = 1$, nor to $|z| > 1$ since we have a pole at 2. Instead, we can use the partial fraction decomposition:

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)} = \frac{(A+B)z - (2A+B)}{(z-1)(z-2)}$$

implies $B = -A$ and so $A = -1$, so

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}.$$

We note that $f_0(z) = \frac{1}{z-2}$ is analytic on $|z| < 2$ and $f_1(z) = -\frac{1}{z-1}$ is analytic on $|z| > 1$, and is analytic at infinity with value 0, so $f(z) = f_0(z) + f_1(z)$ has the desired properties.

In particular, note that the decomposition looks different depending which annulus we take.

Suppose f is analytic on our annulus, so by our proposition we can write $f(z) = f_0(z) + f_1(z)$ with f_0, f_1 satisfying the properties above. Since f_0 is analytic on $|z| < R$, we can write its Taylor series

$$f_0(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Since f_1 is analytic at infinity, with $f(\infty) = 0$, we can write its Taylor series at infinity as

$$f_1(z) = \sum_{n=1}^{\infty} b_n z^{-n} = \sum_{n=-\infty}^{-1} b_{-n} z^n.$$

Putting these together, if we write $a_{-n} = b_n$ for $n \geq 1$, we get

$$f(z) = f_0(z) + f_1(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

for $z \in D$, i.e. an expansion of f as a linear combination of z^n over all integers n, as we claimed we'd find. This is the Laurent expansion of f on D with respect to z_0.

We'd like to have a formula for the coefficients a_n in terms of f. For the Taylor series expansion—equivalently, the Laurent series in the special case where all $a_n = 0$ for $n < 0$—we had

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=R'} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

for some $R' < R$. The first formula doesn't make sense when n is negative, but the second formula actually still does, provided $r < R' < R$, and the same proof applies:

$$\begin{aligned}\frac{1}{2\pi i} \int_{|z-z_0|=R'} \frac{f(z)}{(z-z_0)^{n+1}}\, dz &= \frac{1}{2\pi i} \int_{|z-z_0|=R'} \sum_{m=-\infty}^{\infty} a_m z^{m-n-1}\, dz \\ &= \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty} a_m \int_{|z-z_0|=R'} z^{m-n-1}\, dz,\end{aligned}$$

and the inner integral is 0 unless $m - n - 1 = -1$, i.e. $m = n$, in which case it is $2\pi i$. Hence the whole sum is just a_n as desired, whether n is positive or negative.

In particular, since the coefficients are determined by the values of the function, the Laurent series expansion on a given annulus around a central point is unique (this would also follow from the uniqueness arguments above).

Let's return to our previous example of $f(z) = \frac{1}{(z-1)(z-2)}$ on the most interesting annulus $1 < |z| < 2$. Writing

$$f(z) = f_0(z) + f_1(z) = \frac{1}{z-2} - \frac{1}{z-1},$$

we have

$$f_0(z) = \frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1 - z/2} = \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}} z^n$$

around $z = 0$, and

$$f_1(z) = -\frac{1}{z-1} = -\frac{1/z}{1-1/z} = \sum_{n=1}^{\infty} -z^{-n},$$

so

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

where $a_n = -\frac{1}{2^{n+1}}$ if $n \geq 0$ and $n = -1$ if $n < 0$.

We could also expand f about one of the poles. Consider for example $z_0 = 1$. Writing

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-1} \cdot \frac{1}{(z-1)-1},$$

by the geometric series this is

$$-\frac{1}{z-1}\sum_{n=0}^{\infty}(z-1)^n = \sum_{n=-1}^{\infty} -(z-1)^n$$

for $0 < |z-1| < 1$.