Lecture 18: analytic continuation

Complex analysis, lecture 4
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In the first unit of this class, we were often concerned with what happens to a function as we move along a path. In particular, if the path came back to the same point and had a different value, that meant that the function couldn't be continuous everywhere, which presented a problem; we saw how to repair this sort of defect with branch cuts, and how to view these functions as continuous on a Riemann surface.

We return to this idea, now thinking about analytic functions and with all the machinery we've built up to study them. Now, we know that we can write an analytic function as a power series; so instead of keeping track of just the value of the function as we move along a path, we'll keep track of how the power series expansion changes.

Let's be more precise. Let $\gamma : [a, b] \to D$ be a path in a domain D, and f be an analytic function on D. Set $z_0 = \gamma(a)$ and $z_1 = \gamma(b)$, so that this is a path from z_0 to z_1. Since f is analytic, we can expand it as a Taylor series about any point $\gamma(t)$.

The first thing to observe is that the radius of convergence of f at $\gamma(t)$ continuously depends on $t \in [a, b]$. In fact, the radius of convergence $R(z_0)$ of the power series of f around a point $z_0 \in D$ depends continuously on z_0, so the previous claim follows so long as γ is continuous. Indeed, let z_1 for now be any other point, which we imagine to be close to z_0. Since $R(z_0)$ is the distance from z_0 to the closest point z to z_0 to which f does not extend analytically, and likewise $R(z_1)$ is the distance from z_1 to the closest point z' to z_0 to which f does not extend analytically. Then $R(z_1) = |z_1 - z'| \leq |z_1 - z|$ (since z' is by definition at least as close to z_1 as z) and $|z_1 - z| = |z_1 - z_0 + z_0 - z| \leq |z_1 - z_0| + |z_0 - z| = |z_1 - z_0| + R(z_0)$. Exchanging z_0 and z_1, we likewise have $R(z_0) \leq |z_0 - z_1| + R(z_1)$, so we find that $|R(z_0) - R(z_1)| \leq |z_0 - z_1|$. (Here if both radii are infinite, we say that the difference is 0.) This is the definition of $R(z)$ being a continuous function!

Thus for our path γ, we can consider the disks of convergence along γ, which vary continuously. This lets us define a domain containing z_0 and z_1 on which f is analytic. Without this sort of technique, this may be difficult: the only other natural method is to take a disk around z_0, but we may run into poles of our function before reaching z_1. By choosing our path carefully, under favorable conditions we can avoid the poles to travel from z_0 to z_1.

Let's now back up and make some definitions. If f is analytic on a neighborhood of z_0, we say that it is analytically continuable along γ if for every $t \in [a, b]$, f is analytic at $\gamma(t)$, and for t, t' close enough that the disks on which $f(\gamma(t))$ and $f(\gamma(t'))$ are analytic have some intersection, the corresponding power series agree on these intersections. Then for t and t' close enough that the power series for f near $\gamma(t)$ converges at t', by our results from last time the power series at $\gamma(t')$ is actually determined by that at $\gamma(t)$, since the values of f on a neighborhood of $\gamma(t')$ are determined by this power series. Moving continuously along γ, we see that everything is determined by the power series at the initial point z_0.

We refer to the collection of power series for f at each $\gamma(t)$ as the analytic continuation of f along γ, and the power series at z_1 as the analytic continuation of f to z_1 along γ. Then we have shown the following:

Proposition. *The analytic continuation of f along γ is unique if it exists, and the coefficients $a_n(t)$ of the power expansion of f at $\gamma(t)$ and the radius of convergence $R(\gamma(t))$ depend continuously on t.*

As a corollary, the analytic continuation of f to γ along z_1 depends only on f, z_1, and γ. It is interesting to ask when it is independent of γ.

Let D be a domain, and suppose that f is analytic on D, and γ is a path contained in D. We claim that the analytic continuation of f from z_0 to z_1 is independent of γ. The rough idea is that because f is well-defined and analytic everywhere in D, by some sort of uniqueness theorem we should be able to show that the analytic continuation to z_1 is just the expected, well-defined power series at z_1.

The first thing that comes to mind is the identity theory for Taylor series, which showed that if f and g are analytic on a disk and agree on a smaller disk, then they must agree on the larger disk as well. However, D need not be a disk. Instead, we can use some of the results we proved last time, coming from the study of zeros of analytic functions: we showed that if f and g are analytic on any domain D and agree on some subset with at least one non-isolated point, then $f = g$ on the whole domain D. This now suffices: fix once and for all a choice of, for every $z \in D$, a path γ_z from z_0 to z, with γ_{z_0} the trivial path and $\gamma_{z_1} = \gamma$, and let $g(z)$ be the analytic continuation of f along γ_z to z, which by construction is analytic at z. Then $f = g$ on a neighborhood of z_0, so $f = g$ everywhere, i.e. no matter what paths we choose, the analytic continuation is unique.

However, it is often interesting to look at cases where the analytic continuation *does* depend on the path. This might seem contradictory, since we mostly care about analytic functions, but recall the setup for branch cuts: if we take a path from z_0 to itself which crosses a branch cut, then we expect that f at the start does not agree with f at the end.

Consider the example $f(z) = \sqrt{z}$, the principal branch of the square root; more precisely, $f(re^{i\theta}) = \sqrt{r}e^{i\theta/2}$, making a branch cut along some ray from the origin.

Consider the path $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$, from $z_0 = 1$ to itself in a loop around the origin. At $z_0 = \gamma(0)$, f is analytic, with series expansion

$$f(z) = f_0(z)t = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \frac{1}{16}(z-1)^3 - \frac{5}{128}(z-1)^4 + \cdots$$

(there exists a precise formula for the coefficients, but it is a little annoying to write down so we stick with the first few terms by computing derivatives). We can likewise find the power expansion at $z = e^{i\theta}$, which we write as

$$f_t(z) = e^{i\theta/2} + \frac{1}{2e^{i\theta/2}}(z-e^{i\theta}) - \frac{1}{8e^{3i\theta/2}}(z-e^{i\theta})^2 + \frac{1}{16e^{5i\theta/2}}(z-e^{i\theta})^3 - \frac{5}{128e^{7i\theta/2}}(z-e^{i\theta})^4 + \cdots .$$

Therefore at $\theta = 2\pi$ we find

$$f_{2\pi}(z) = -1 - \frac{1}{2}(z-1) + \frac{1}{8}(z-1)^2 - \frac{1}{16}(z-1)^3 + \frac{5}{128}(z-1)^4 + \cdots = -f_0(z)$$

is the opposite branch of the square root, just like we found when studying branches and phase factors. Note that this does not contradict the result above because there is no way to make f analytic on all of $\mathbb{C}$, or even any domain in $\mathbb{C}$ containing γ: in order for f to be continuous we have to exclude some branch cut from 0 to ∞, which will necessarily cross the path γ.

Even if f is not known to be analytic on all of D, as in this example, just as for branch cuts or integrals we can deform the paths: just as for the deformation theorems, if we can continuously vary between two paths γ_0 and γ_1 while holding the endpoints z_0, z_1 constant, then the analytic continuation of f to z_1 along γ_0 and along γ_1 agree. This is sometimes called the monodromy theorem, and can be useful for calculating analytic continuations along weird paths, by deforming them to simpler ones.

More generally, if f is analytic away from some isolated set of singularities, at any fixed point z_0 we can only define f by a power series at z_0 in a disk of finite radius; sometimes we will talk about the analytic continuation of f to $\mathbb{C}$ (other than the singularities) as the (necessarily unique, as above!) analytic function on $\mathbb{C}$ whose restriction to the disk is this power series. A simple example is the geometric series

$$f(z) = \sum_{n=0}^{\infty} z^n$$

near $z_0 = 0$. If we define this to be our function, it is only well-defined on $|z| < 1$. However, when $|z| < 1$ we know that it satisfies the rule

$$(1 - z)f(z) - 1 = 0,$$

since in fact $f(z) = \frac{1}{1-z}$, so by the permanence principle for functional equations we know that if f extends to an analytic function on $\mathbb{C}$ (minus some isolated set of points), it must still satisfy this equation, i.e.

$$f(z) = \frac{1}{1-z}$$

is the unique extension of f to $\mathbb{C} \setminus \{1\}$. Note that if we plugged in some z with $|z| \geq 1$ into the original definition, we would still get something ill-defined: e.g. at $z = -1$, we have

$$f(z) = \frac{1}{1-(-1)} = \frac{1}{2} \text{ “ = ” } \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots ,$$

which plainly does not converge. This is one technique sometimes used to assign values to divergent sums. Indeed, in the example above the partial sums alternate between 0 and 1, so in some sense it is reasonable to say that if it were to converge, the right value would be $\frac{1}{2}$.

However, if we were to plug in e.g. $z = 2$, we would get by this method

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \cdots \text{ “ = ” } \frac{1}{1-2} = -1,$$

which is plainly absurd, so don't take this method too seriously. It is also not hard to find different functions f obtaining the same sums by analytic continuation. Indeed, even for the first, relatively reasonable-looking example above, consider

$$g(z) = \frac{1+z}{1+z+z^2} = \frac{1}{1-z^3} - \frac{z^2}{1-z^3} = \sum_{n=0}^{\infty}(z^{3n} - z^{3n+2}) = 1 - z^2 + z^3 - z^5 + z^6 - z^8 + \cdots$$

which converges for $|z| < 1$. If we were to evaluate at $z = 1$ via the method above, we would get

$$g(1) = \frac{2}{3} \text{ “} = \text{”} \ 1 - 1 + 1 - 1 + \cdots ,$$

assigning a different value to the same divergent sum.

It is however a very important method for defining functions on $\mathbb{C}$ minus an isolated set, or more generally on larger domains, which might initially be defined by series—or even integrals—convergent on a smaller region. Another important example is the gamma function

$$\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}\, dt,$$

which converges when s has real part greater than 0. By integration by parts, one can show

$$\Gamma(s) = (s-1)\Gamma(s-1),$$

so in turn $\Gamma(s-1) = \frac{1}{s-1}\Gamma(s)$. This lets us define Γ on points with negative real part: for example, $\Gamma(-1/2)$, not a priori defined, is given by the above formula with $s = \frac{1}{2}$: $\Gamma(-1/2) = \frac{1}{-1/2}\Gamma(1/2) = -2\Gamma(1/2)$.

One can calculate $\Gamma(1) = 1$, so by the iterative rule $\Gamma(2) = 1 \cdot 1 = 1$, $\Gamma(3) = 2 \cdot 1 = 2$, $\Gamma(4) = 3 \cdot 2 = 6$, and so on: by induction we see that $\Gamma(n) = (n-1)!$. However, at $s = 0$ there is a pole, so by the rule above Γ also has a pole at every negative integer. Other than at these points, though, this lets us extend Γ to an analytic function on $\mathbb{C} \setminus \{0, -1, -2, ...\}$.