Lecture 17: series expansions at infinity and zeros

Complex analysis, lecture 4
October 22, 2025

1. Power expansions at infinity

We return to the perspective of the Riemann sphere, the complex numbers plus a point ∞. We've seen before that thinking of this extension makes some statements nicer. Today, we want to unify this perspective with our new tools for studying analytic functions.

If g is a function defined on some disk D centered at 0, we define $f(z) = g(1/z)$, which will then be defined for $|z|$ sufficiently *large* (equivalently $|1/z|$ sufficiently small), and we say that f is analytic at infinity if g is analytic at 0. More broadly, we refer to a domain $D \subset \mathbb{C}$ as a "neighborhood of ∞" if for R large enough, $|z| > R$ implies that $z \in D$. Note that this is equivalent to the set $\{z : \frac{1}{z} \in D\}$ containing $\{z : |z| < 1/R\} \setminus \{0\}$, i.e. being a (punctured) neighborhood of 0. Given a function f on D, we say it is analytic at ∞ if $g(z) = f(1/z)$ is analytic at 0.

It is often useful to make a change of variables: set $w = \frac{1}{z}$, so that studying what happens near $z = \infty$ is equivalent to studying what happens near $w = 0$, with which we are more familiar. We will sometimes write $f(\infty) = \lim_{z\to\infty} f(z) = \lim_{w\to 0} g(w) = g(0)$, when the limits exist.

For example, consider $f(z) = \frac{z}{z+1}$, defined on $|z| > 1$ (or more generally on $\mathbb{C} \setminus \{-1\}$). Then $g(w) = f(1/w) = \frac{1/w}{1/w+1} = \frac{1}{1+w}$ is analytic at $w = 0$ (in fact, at all $w \neq -1$), so f is analytic at ∞. We have

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{z}{z+1} = 1 = \lim_{w\to 0} g(w) = g(0).$$

On the other hand, a function like $f(z) = e^z$, while well-behaved everywhere in $\mathbb{C}$, is *not* analytic at ∞. Indeed, $g(w) = f(1/w) = e^{1/w}$ is not analytic at 0. More generally, a necessary—but not sufficient—condition for f to be analytic at infinity is that the limit

$$\lim_{z\to\infty} f(z) = \lim_{w\to 0} g(w)$$

exist, which it does not in this case. (We might call this condition being continuous at infinity, though this by itself won't often come up.)

(Note in the above that the limit must exist as a complex number; the limit being infinite does not suffice. However, there does exist a notion of analytic functions $\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ for which the limit being infinite, properly defined, would suffice, which we may come back to later.)

If $g(w)$ is analytic at 0, we have seen that for $|z| < R$ for some radius $0 \leq R \leq +\infty$, we can write

$$g(w) = \sum_{n=0}^{\infty} a_n w^n = a_0 + a_1 w + a_2 w^2 + \cdots$$

where $a_n = \frac{1}{n!} g^{(n)}(0)$. Rewriting everything in terms of f and z, this gives us the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^{-n} = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots ,$$

which we refer to as the series or Taylor expansion of f at infinity.

This is perhaps very strange-looking, but should be expected from the terminology: we have argued before that if f is analytic at a point, then f admits a Taylor expansion centered at that point; this is the extension of this principle to ∞.

What about the convergence of this series? We can study its convergence in terms of that of $g(w)$: the series for $g(w)$ should converge absolutely to an analytic function of w for $|w| < R$ for some R, so the series for f should converge for $|z| > \frac{1}{R}$. If $R = 0$, then the series converges nowhere; if $R = +\infty$, i.e. g is entire, then the series for f also converges everywhere except at $z = 0$ (where it is undefined, though f may extend to this point).

Let's return to the example above. If $f(z) = \frac{z}{z+1}$, we saw $g(w) = \frac{1}{1+w}$, which has power expansion given by the geometric series in $-w$,

$$g(w) = \sum_{n=0}^{\infty} (-1)^n w^n$$

for $|w| < 1$. Therefore

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^{-n}$$

for $|z| > 1$. Indeed, the series on the right can be viewed as the geometric series in $-1/z$, and we can check explicitly that this recovers $f(z)$.

2. Zeros of analytic functions

Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

We want to study the zeros of f.

In general terms, these could be anything. There is one point which is natural to study, though: at $z = z_0$, all higher terms vanish and we find $f(z_0) = a_0$. Therefore f has a zero at z_0 if and only if $a_0 = 0$, in which case

$$f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots .$$

Now, in the real setting, for a function like $f(x) = x^2$ we say that f has a "double zero" at $x = 0$, since there are in some sense two factors which both vanish at 0. This makes statements like the fundamental theorem of algebra work: "every polynomial of degree d has exactly d zeros in $\mathbb{C}$" is only true if we count multiple zeros, otherwise e.g. $f(z) = z^2$ would be a counterexample, having only one zero.

We make this definition precise here: we say that f has a double zero at z_0, or a zero of order 2, if $a_0 = a_1 = 0$, i.e.

$$f(z) = a_2(z - z_0)^2 + a_3(z - z_0)^3 + a_4(z - z_0)^4 + \cdots .$$

More generally, we say that f has a zero of order n at z_0 if $a_0 = a_1 = \cdots = a_{n-1} = 0$, so that

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots .$$

For $n = 0$, this is just $f(z) = a_0 + a_1(z - z_0) + \cdots$, so we sometimes refer to a point z_0 at which $f(z_0) \neq 0$ as a "zero of order 0." For $n = 1$, we sometimes call a zero of order 1 a "simple zero."

All this was special to the point z_0 at which we took our Taylor expansion. However, note that we can take our expansion about any point z_0 at which f is analytic; and the coefficients a_n of the expansion at that point are given by $\frac{1}{n!} f^{(n)}(z_0)$. Therefore we make the following definition: if $f : D \to \mathbb{C}$ is analytic on D, for every $z_0 \in D$ we say that f has a zero of order n at z_0 if $f^{(k)}(z_0) = 0$ for $0 \leq k \leq n - 1$. By the expansion above, if f has a zero of order n at z_0, then we can write

$$f(z) = (z - z_0)^n h(z)$$

for some function h which is analytic at z_0 with $h(z_0) \neq 0$.

For example, consider $f(z) = \sin z$, which is analytic on all of $\mathbb{C}$. Take $z_0 = \pi$. We have $f(\pi) = 0$, $f'(\pi) = \cos \pi = -1$, $f''(\pi) = -\sin \pi = 0$, and $f'''(\pi) = -\cos \pi = 1$, after which the derivatives repeat, so the Taylor expansion at π is given by

$$\sin z = -(z - \pi) + \frac{1}{6}(z - \pi)^3 - \frac{1}{120}(z - \pi)^5 + \cdots .$$

In particular, $\sin z$ has a zero of order 1 at π. Meanwhile $g(z) = \sin z + z - \pi = \frac{1}{6}(z - \pi)^3 - \cdots$ has a zero of order 3 at π.

If f has a zero of order n at z_0 and g has a zero of order m at z_0, then fg has a zero of order $n + m$. Indeed, write

$$f(z) = (z - z_0)^n h(z), \qquad g(z) = (z - z_0)^m j(z);$$

then

$$f(z)g(z) = (z - z_0)^{n+m} h(z)j(z),$$

and $h(z)j(z)$ is analytic and nonzero at z_0 since each factor is.

We can understand the case at infinity discussed above, too: if f is analytic at infinity, we say it has a zero of order n at infinity if $g(w) = f(1/w)$ has a zero of order n at 0, or equivalently if the series expansion at infinity is of the form

$$f(z) = \frac{a_n}{z^n} + \frac{a_{n+1}}{z^{n+1}} + \cdots = \frac{1}{z^n}\left(a_n + \frac{a_{n+1}}{z} + \cdots\right),$$

i.e. if $f(z) = \frac{1}{z^n}h(z)$ where h is analytic at infinity and $h(\infty) \neq 0$.

For example, consider $f(z) = \frac{1}{z^2+1}$. We have $g(w) = f(1/w) = \frac{1}{w^{-2}+1} = \frac{w^2}{1+w^2}$, which has a double zero at $w = 0$: its power expansion is

$$w^2 - w^4 + w^6 - w^8 + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}w^{2n}.$$

Therefore f has a double zero at infinity. Its series expansion at infinity is

$$z^{-2} - z^{-4} + z^{-6} - \cdots = \frac{1}{z^2}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \cdots\right).$$

For any set $S \subset \mathbb{C}$, we say that a point $z_0 \in S$ is isolated if there exists some $r > 0$ such that for every $z_0 \neq z \in S$, $|z - z_0| \geq r$. Thus for example in an interval $[a, b] \subset \mathbb{R} \subset \mathbb{C}$, no point is isolated, while in any finite collection of points every point is isolated.

Theorem. *If $D \subset \mathbb{C}$ is a domain, $f : D \to \mathbb{C}$ is an analytic function, and $S = \{z \in D : f(z) = 0\}$, then either $f(z) = 0$ for all $z \in D$ (i.e. $S = D$) or every point in S is isolated.*

Proof. First, assume that every $z_0 \in S$ has finite order, so there exists some positive integer n such that $f(z) = (z - z_0)^n h(z)$ with $h(z)$ analytic on some disk around z_0 with $h(z_0) \neq 0$. Since h is analytic it is continuous, so if we pick z sufficiently close to z_0 then $h(z)$ must also be nonzero, and so $f(z) = (z - z_0)^n h(z)$ is nonzero for z sufficiently close to z_0. Therefore z_0 is isolated in S, since every point within a certain radius of z_0 is not in S.

It remains to show that every zero z_0 has finite order, equivalently that at least one of the derivatives $f^{(n)}(z_0)$ is nonzero. Let $U \subset D$ be the subset of points z_0 at which $f^{(n)}(z_0) = 0$ for all n, and assume it is nonempty, with $z_0 \in U$. Then on a disk of some radius centered at z_0, f is equal to its Taylor expansion around z_0, which is $0 + 0 + \cdots = 0$, so f is identically zero in a neighborhood of z_0, hence every point in this disk is in U. Therefore U is an open set: it contains a disk around every point it contains.

On the other hand, if $z_0 \notin U$, so some higher derivative is nonzero, then by the argument above we can find a sufficiently small disk around z_0 where the function is nonzero, so the disk around z_0 is also not contained in U. Hence $D \setminus U$ is also open. Since D is connected, the only way this is possible is if U is empty (i.e. every zero is of finite order, showing as above that every zero is isolated) or if $U = D$ (in which case $f = 0$ on D). ☐

As a corollary, we deduce the following.

Corollary. *Let D be a domain and $f, g : D \to \mathbb{C}$ analytic functions on D. If $S \subset D$ is a set with a non-isolated point and $f(z) = g(z)$ for $z \in S$, then $f = g$ on D.*

The proof is by applying the above theorem to $f - g$.

An important case of the above uniqueness principle is when D contains $\mathbb{R}$, or some interval in $\mathbb{R}$, on which f and g agree. For example, consider $f(z) = (\sin z)^2 + (\cos z)^2$. We know that if $z \in \mathbb{R}$, then $f(z) = 1$. Taking $g(z) = 1$, observe that $\mathbb{R}$ contains non-isolated points (in fact all its points are non-isolated), hence since $f = g$ on $\mathbb{R}$ we must have $f = g$ on all of $\mathbb{C}$, i.e. the functional equation $\sin^2 z + \cos^2 z = 1$ extends to $\mathbb{C}$.

This can be generalized by the following permanence principle for functional equations:

Proposition. *Let $D \subset \mathbb{C}$ be a domain, $S \subset D$ a subset with a non-isolated point, and $F(z, w)$ a function on $z, w \in D$ such that for each fixed z_0, $F(z_0, w)$ is an analytic function of w, and likewise for each fixed w_0, $F(z, w_0)$ is an analytic function of z. If $F(z, w) = 0$ for $z, w \in S$, then $F(z, w) = 0$ for all $z, w \in D$.*

Indeed, fixing $z_0 \in S$, we have an analytic function $w \mapsto F(z_0, w)$ whose restriction to S is zero, so it must be zero on all of D by the above. Therefore $F(z, w) = 0$ for $z \in S$ and $w \in D$; fixing $w_0 \in D$, $F(z, w_0)$ is then an analytic function on D whose restriction to $z \in S$ is zero, so it must be zero for all $z \in D$ as well.

As an application, we can prove the relation $e^{z+w} = e^z e^w$, which previously we saw by hand. Recall we mentioned that one approach to the class would be to introduce the complex exponential by its Taylor series and then prove it has the usual properties; this is the key property we'd want to prove. Taking $F(z, w) = e^{z+w} - e^z e^w$ on $z, w \in \mathbb{C}$, we know that $F(z, w) = 0$ for $z, w \in \mathbb{R}$; so it must in fact vanish everywhere, i.e. $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.