Lecture 16: Taylor series

Complex analysis, lecture 4

October 17, 2025

Last time, we studied power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for sequences a_n, and saw that within a disk of some radius (possibly zero or infinite) centered at z_0, this series converges absolutely to an analytic function of z. Our first goal today is to prove the converse, justifying our terminology for analytic functions.

Theorem. *Suppose that f is an analytic function on $D = \{z : |z - z_0| < R\}$ for some R. Then:*

(a) for all $|z - z_0| < R$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{n!} f^{(n)}(z_0),$$

and the power series has radius of convergence greater than or equal to R.

(b) For any fixed $0 < r < R$ and $n \geq 0$,

$$a_n = \frac{1}{2\pi i} \oint_{|z - z_0| = r} \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$

(c) For any fixed $0 < r < R$ and $n \geq 0$, if $|f(z)| \leq M$ for all $|z - z_0| = r$, then

$$|a_n| \leq \frac{M}{r^n}.$$

Proof. The idea is to look at Cauchy's formula: for $|z - z_0| < r < R$,

$$f(z) = \frac{1}{2\pi i} \oint_{|w - z_0| = r} \frac{f(w)}{w - z} \, dz$$

since $|w - z_0| < r$ is a domain containing z, and write

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \cdot \frac{1}{1 - (z - z_0)/(w - z_0)} = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n$$

(since $|w-z| = r \neq 0$ and $\frac{|z-z_0|}{|w-z_0|} = \frac{|z-z_0|}{r} < 1$). This can be rewritten as

$$\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}}.$$

Plugging this in to Cauchy's formula, we get

$$f(z) = \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(w) \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}}\, dz.$$

By uniform convergence within the disk of radius $r < R$, we can exchange the sum (viewed as a limit of partial sums) and the integral, so this is

$$\frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_{|z-z_0|=r} f(w) \frac{(z-z_0)^n}{(w-z_0)^{n+1}}\, dw = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}}\, dw \right) (z-z_0)^n.$$

If we label the quantity in parentheses as a_n, as in (b), by Cauchy's integral formula for derivatives this agrees with

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

as in (a), so we have proven

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for both descriptions. This converges for $|z-z_0| < r$ for any $r < R$, hence for any $|z-z_0| < R$, so its radius of convergence is at least R. Finally (c) follows from the Cauchy estimates. □

Thus we have shown that analytic functions are equal to their Taylor expansions, at least on some sufficiently small disk. An immediate consequence is the following uniqueness statement:

Corollary. *Let f, g be analytic functions on $\{z : |z-z_0| < r\}$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \geq 0$. Then $f(z) = g(z)$ for all $|z-z_0| < r$.*

Indeed, both functions are equal to their Taylor series on this region, which are equal by assumption.

We can reframe this slightly, which will be useful later on:

Corollary. *Let $0 < r < R$ be real numbers, and f, g analytic functions on $D = \{z : |z-z_0| < R\}$ such that if $|z-z_0| < r$, then $f(z) = g(z)$. Then $f(z) = g(z)$ for all $|z-z_0| < R$.*

Indeed, by assumption $f = g$ on the smaller disk, so their derivatives agree there, and in particular $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all n; so since f and g are equal to their Taylor series on D, they must be equal there by the previous corollary.

We now turn to some examples. Consider the function $f(z) = e^z$. Since $f^{(n)}(z) = e^z$ for all n and f is entire, we have for all $z \in \mathbb{C}$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

Note that although this is a direct consequence of the above statements, this is actually rather remarkable: in order to talk about the complex exponential, we defined its extension to complex numbers in an apparently arbitrary way; and although we saw some reasons to believe this was the right definition, we can now prove that this is the only possible one: the only analytic function f such that $f'(z) = f(z)$ and $f(0) = 1$, a differential equation which should have unique solution e^z, is given by this Taylor series. An alternate approach would have been to define this to be the complex exponential, and deduce all its properties from there.

As an example of this approach, we deduce Euler's formula. Recall the Taylor series for sine and cosine (by the same argument, the same as their real analogues, and convergent on the whole complex plane since they are entire):

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{1}{2} z^2 + \frac{1}{24} z^4 - \cdots,$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{1}{6} z^3 + \frac{1}{120} z^5 - \cdots.$$

It follows that

$$\begin{aligned} \cos(z) + i\sin(z) &= 1 + iz - \frac{1}{2} z^2 - \frac{1}{6} iz^3 + \frac{1}{24} z^4 + \frac{1}{120} iz^5 - \cdots \\ &= 1 + iz + \frac{1}{2} i^2 z^2 + \frac{1}{6} i^3 z^3 + \frac{1}{24} i^4 z^4 + \frac{1}{120} i^5 z^5 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (iz)^n \\ &= e^{iz}. \end{aligned}$$

The special case with which we're familiar is when $z = \theta$ is real, giving points $e^{i\theta} = \cos\theta + i\sin\theta$ on the unit circle, but one could also take z complex, recovering our formulas for sine and cosine of complex numbers, which agree with the power series above.

Another more abstract corollary of our result is the following reinterpretation of the radius of convergence:

Proposition. *If $f(z)$ is analytic at z_0 with Taylor series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

this power series has radius of convergence given by the largest real number R (or $+\infty$) such that f extends to an analytic function on $\{z : |z - z_0| < R\}$.

Note that by the corollary above, such an extension is unique, so this really is just a property of f with no need to worry about the choice of an extension.

For example, the geometric series $\sum_{n=0}^{\infty} z^n$ is the Taylor series for $f(z) = \frac{1}{1-z}$ centered at $z_0 = 0$. This function has a pole at $z = 1$, so the largest disk it extends to is radius 1, hence it has radius of convergence 1.

A more subtle example is $f(z) = \frac{1}{z^2+1}$. In the real setting, this is well-defined and analytic on all of $\mathbb{R}$; at $z = 0$, it has the power expansion

$$\frac{1}{z^2+1} = \sum_{n=0}^{\infty}(-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots ,$$

given by taking the geometric series in $-z^2$. However, this only converges for $|z| < 1$, somewhat inexplicably from what we can see over $\mathbb{R}$: there are no singularities to worry about.

In the complex setting, however, we can see what's happening more clearly: here (by the fundamental theorem of algebra) $\frac{1}{z^2+1}$ does have singularities, at $z = \pm i$. These are at distance 1 from the origin, so the Taylor series can converge only within the disk of radius 1. This is an instance of the following principle, which we will see more of in the future: to understand analytic properties of real functions of a real variable, especially those which extend to analytic functions of a complex variable, it is often useful to pass into the complex plane even when complex numbers don't a priori appear.

Another example of interest is the case of removable singularities: consider $f(z) = \frac{z^2-1}{z+1}$. Near $z = 0$, this is the same thing as $z - 1$; but at $z = -1$ it is undefined. Nevertheless, studying the Taylor series at $z = 0$, we find that it is $-1 + z$, a finite polynomial since f is here, which necessarily converges everywhere. Indeed, f extends (uniquely) to an analytic function on all of $\mathbb{C}$, even though a priori it isn't defined at $z = -1$, with extension $z - 1$, so the proposition implies that the radius of convergence should be infinite.

Today's theorem on Taylor series is the key property of power series (and, arguably, of analytic functions); next week we'll study some complements such as power expansions at infinity, zeros of analytic functions, and analytic continuation.