Lecture 15: power series

Complex analysis, lecture 4
October 8, 2025

1. More on uniform convergence

Last time, we recalled the notions of convergence of series and uniform convergence of sequences of functions. In particular, we noted that the uniform limit of continuous functions is continuous, so uniform convergence seems to be better suited to the study of analytic properties of functions than pointwise convergence.

Today, we make some more observations along these lines, now targeting analytic functions.

Proposition. *Let γ be a piecewise smooth path in $\mathbb{C}$ and $\{f_n\}$ be a sequence of continuous functions on γ converging uniformly to f. Then*

$$\lim_{n\to\infty}\int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz.$$

This follows from the ML bound: we can find $\epsilon_n \to 0$ such that $|f(z) - f_n(z)| < \epsilon_n$ for every $z \in \gamma$, so if γ has length L then

$$\left|\int_\gamma f(z)\,dz - \int_\gamma f_n(z)\,dz\right| = \left|\int_\gamma (f(z) - f_n(z))\,dz\right| \le L\epsilon_n \to 0$$

as $n \to \infty$.

As a corollary, we obtain the following analytic version of our theorem from last time, which is really what we want for this class:

Theorem. *If $\{f_n\}$ is a sequence of analytic functions on a domain D converging uniformly to f, then f is analytic.*

Proof. By Cauchy's formula,

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f_n(w)}{w-z}\,dw,$$

so taking the limit by the previous proposition gives

$$f(z) = \lim_{n\to\infty} f_n(z) = \frac{1}{2\pi i}\int_{\partial D}\lim_{n\to\infty}\frac{f_n(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{w-z}\,dw,$$

which is an analytic formula.

An alternative proof is by Morera's theorem (which avoids concerns about some missing hypotheses above on D for Cauchy's theorem!). First, note that since the f_n are analytic, they are continuous, so f is continuous. Let $R \subset D$ be any rectangle as in Morera's theorem. By Cauchy's theorem and the proposition above,

$$\int_{\partial R} f(z)\,dz = \lim_{n\to\infty} \int_{\partial R} f_n(z)\,dz = 0,$$

so by Morera's theorem f is analytic on D. □

The proof by Morera's theorem is stronger in that it lets us avoid putting additional hypotheses on D (e.g. bounded, with the f_n extending to the boundary). However, the proof by Cauchy's formula extends nicely to higher derivatives: on suitable D, if the f_n are analytic and converge uniformly to f, then $f_n^{(m)}$ converge uniformly to $f^{(m)}$.

2. Power series

Fix $z_0 \in \mathbb{C}$. A power series centered at z_0 is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for complex numbers a_n. By making the change of variables $w = z - z_0$, we may as well assume $z_0 = 0$ to simplify the notation, and often do so. (Of course, we could also change the indexing to start at any nonnegative integer, corresponding to taking the first few terms to be zero; but we cannot have negative indices!)

This series may or may not converge at each point z; it always converges at $z = z_0$, since then every term except the constant $n = 0$ term necessarily vanishes. More generally, we have the following result.

Theorem. *For any power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, there exists $R \in [0, \infty) \cup \{+\infty\}$ such that if $|z - z_0| < R$, the series converges absolutely; and if $|z - z_0| > R$, it diverges. For each $r < R$, the series converges uniformly on the closed disk $\{z : |z - z_0| \le r\}$.*

This R is called the radius of convergence of the series. Here we say that every real number is strictly less than $+\infty$.

Notably the series does not necessarily converge uniformly on the open disk $\{z : |z - z_0| < R\}$. For example, for the power series $\sum_{n=0}^{\infty} z^n$, the radius of convergence is 1, so the series converges to $\frac{1}{1-z}$ whenever $|z| < 1$; but the difference

$$\frac{1}{1-z} - \sum_{k=0}^{n} z^n = \sum_{k=n+1}^{\infty} z^n = \frac{z^{n+1}}{1-z}$$

grows without bound as $z \to 1$. However, if we restrict to $|z| \le r$ for some $r < 1$, then this is uniformly bounded in absolute value, by

$$\frac{|z|^{n+1}}{1 - |z|} \le \frac{r^{n+1}}{1 - r}.$$

Note also that the theorem has nothing to say about the convergence of the power series on the circle $\{z : |z - z_0| = R\}$: here it may converge absolutely, conditionally, or diverge, depending on the series in question. For example, the geometric series diverges for every z with $|z| = 1$, but a mild modification

$$\sum_{n=1}^{\infty} \frac{1}{n} z^n$$

converges conditionally at $z = -1$ (by the alternating series test) and diverges at $z = 1$ (where it is the harmonic series, and the further modification

$$\sum_{n=1}^{\infty} \frac{1}{n^2} z^n$$

converges uniformly on the closed unit disk (one can bound the terms by $\frac{1}{n^2}$, whose sum converges by e.g. the integral test). All of these have radius of convergence 1, but different behavior on the boundary circle.

Most of this theorem should be familiar from calculus, except possibly the part about uniform convergence, which can be proven using the Weierstrass M-test.

For completeness, we mention some examples with radius of convergence different from 1: an easy example is something like

$$\sum_{n=0}^{\infty} \frac{z^n}{2^n}.$$

This is the geometric series for $\frac{z}{2}$, which converges for $|z/2| < 1$, i.e. for $|z| < 2$, so the radius of convergence is 2. For something like the Taylor series

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n,$$

the series converges (absolutely) everywhere, so the radius of convergence is $+\infty$. In the other direction,

$$\sum_{n=0}^{\infty} n! z^n$$

does not converge for any z with $|z| > 0$, so its radius of convergence is 0.

Recall that a series is defined to be the limit of its partial sums. We are now thinking of these as functions of a complex variable z, so we can ask about various properties. In particular, the partial sums of power series are polynomials, which are analytic everywhere. By the theorem above, for any $r < R$ the series converges uniformly to a function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

on $\{z : |z| \leq r\}$, where we've taken $z_0 = 0$, and so f is an analytic function on this region. Letting r vary, we've proven the following:

Theorem. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

have radius of convergence R*. Then* f *is analytic on* $\{z : |z| < R\}$*.*

We can then differentiate both sides, and on this region everything is as expected by uniform convergence: the derivative is term-by-term, i.e.

$$f'(z) = \sum_{n=1}^{\infty} a_n n z^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1) z^n.$$

Evaluating at $z = 0$, we find $f(0) = a_0$, $f'(0) = a_1$. Repeating for higher derivatives gives the identity

$$f^{(n)}(0) = a_n n!,$$

or

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

More generally for arbitrary z_0, if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

then

$$a_n = \frac{f^{(n)}(z_0)}{n!},$$

and so f is equal to its Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

for $|z - z_0| < R$. In other words, we have shown that a function which is locally equal to its Taylor series—the definition of analytic functions from calculus or real analysis—is analytic in our sense (or holomorphic to avoid confusion). Next time, we will see the converse, which is one of the most important theorems we have claimed but not proven: a holomorphic function is analytic in the Taylor series sense.

One can also integrate functions term by term using their Taylor series, provided we restrict to disks of radius strictly smaller than R. Integration and differentiation can be used to obtain new series from old ones. For example, differentiating the geometric series gives

$$\frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n = 1 + 2z + 3z^2 + \cdots.$$

Finally, we mention two formulae for determining the radius of convergence, based on the ratio test and the root test respectively: if the limits

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

or

$$\lim_{n\to\infty}|a_n|^{-1/n}$$

exist, either as (necessarily nonnegative) real numbers or as $+\infty$, then they are equal to R.

Note that it is possible for one or both limits to fail to exist even when R is well-defined (as it is for every power series). For example, consider the power series

$$\sum_{n=0}^{\infty} z^{2n}.$$

We can understand this very simply as the geometric series for z^2, so its radius of convergence is 1. However, as a power series it corresponds to $a_n = 0$ if n is odd and $a_n = 1$ if n is even; so in each formula the values are undefined in general, even after truncating a finite number of the leading terms.

There is however a more general formula which always exists:

$$R = \frac{1}{\limsup_{n\to\infty}|a_n|^{1/n}}.$$

For the example above, $\limsup_{n\to\infty}|a_n|^{1/n} = 1$, and so we recover the right radius of convergence.