Lecture 14: series and convergence

Complex analysis, lecture 4
October 8, 2025

1. Infinite series

Our goal for today will be to go over some facts about infinite series and sequences of functions. Just like with our review of integration last unit, much of this will be familiar from calculus or real analysis, but we'll focus on aspects that will be relevant in the complex setting, and see some new properties. Most of the behavior really unique to the complex setting will wait until next week however.

For a sequence $a_0, a_1, a_2, \dots$ (say of complex numbers) and $n \geq 0$, we can define the partial sum

$$S_n = \sum_{k=0}^{n} a_k.$$

We define the infinite series

$$\sum_{k=0}^{\infty} a_k := \lim_{n\to\infty} S_n,$$

so that the series converges if (and only if) the limit of the partial sums does.

Since it is often difficult to give a precise formula for the S_n, it is useful to have criteria for when a series converges or diverges. One such test is as follows: if $\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} S_n$ converges, then writing $a_n = S_n - S_{n-1}$ for $n \geq 1$ we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = 0,$$

so conversely if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{k=0}^{\infty}$ must diverge.

This is a rather weak condition: there are many series whose terms tend to 0 but which still do not converge. Consider for example

$$\sum_{k=1}^{\infty} \frac{1}{k}.$$

The partial sums

$$\sum_{k=1}^{n} \frac{1}{k}$$

are *lower* bounded by

$$\sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{1}{x}\, dx = \log(n+1).$$

Since the right-hand side tends to infinity, the left-hand side cannot be bounded, so the series diverges.

We therefore need a new, more sensitive test. Suppose $\{a_n\}$ is a sequence of nonnegative real numbers, so $S_n = \sum_{k=0}^n a_k$ is nondecreasing: $S_{n+1} = S_n + a_{n+1} \geq S_n$. A nondecreasing sequence of real numbers converges if and only if it's bounded above, so $\sum_{k=0}^\infty a_k$ converges if and only if the partial sums are bounded above.

In particular, if $\{b_n\}$ is another sequence of nonnegative real numbers such that $0 \leq a_n \leq b_n$ for every n and $\sum_{k=0}^\infty b_k$ converges, that means that the partial sums $\sum_{k=0}^n b_k$ are bounded, so

$$\sum_{k=0}^{n} a_n \leq \sum_{k=0}^{n} b_n$$

are also bounded, hence $\sum_{k=0}^\infty a_k$ converges. This is the comparison test, which we restate for emphasis:

Proposition (Comparison test)**.** *If $0 \leq a_n \leq b_n$ are sequences of real numbers and*

$$\sum_{k=0}^{\infty} b_k$$

converges, then so does

$$\sum_{k=0}^{\infty} a_k.$$

Taking the contrapositive, this means that if $0 \leq a_n \leq b_n$ and $\sum_{k=0}^\infty a_k$ diverges, then so does $\sum_{k=0}^\infty b_k$.

In order to make use of this test, we need to have some series which we know converge or diverge. We have one example of a divergent series above. An example of a convergent series is given by the geometric series

$$\sum_{k=0}^{\infty} z^k$$

for $|z| < 1$. Indeed, it is a standard algebraic fact that

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

for $z \neq 1$, and if $|z| < 1$ then $\lim_{n\to\infty} z^{n+1} = 0$ so the series converges to

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}.$$

This is a very important series which we will see frequently.

There are of course many other convergence tests which I won't go over now; these may come up, and you will be expected to know anything that would typically show up in a calculus class, but they will not be a focus.

Another useful fact, which starts to exhibit some of the complex behavior, is absolute convergence:

Proposition. *Let $\{a_n\}$ be a sequence of complex numbers. If*

$$\sum_{n=0}^{\infty} |a_n|$$

converges, then so does

$$\sum_{n=0}^{\infty} a_n,$$

and

$$\left|\sum_{n=0}^{\infty} a_n\right| \leq \sum_{n=0}^{\infty} |a_n|.$$

This is a standard calculus result for real sequences, but notice that the absolute value in the complex setting is a little more complicated so we need to be a little careful.

Proof. It suffices to prove that the sums of $\mathrm{Re}(a_n)$ and $\mathrm{Im}(a_n)$ converge. The idea is to turn these into sequences of positive real numbers and use the comparison test. The key observation is $|\mathrm{Re}(z)| < |z|$, so $0 \leq \mathrm{Re}(z) + |z| \leq 2|z|$. For $z = a_k$, this tells us that on the one hand

$$\sum_{k=0}^{n} (\mathrm{Re}(a_k) + |a_k|)$$

is on the one hand a sum of nonnegative real terms and on the other hand is upper bounded by

$$2\sum_{k=0}^{n} |a_k|,$$

which converges by assumption. Therefore

$$\sum_{n=0}^{\infty} (\mathrm{Re}(a_n) + |a_n|)$$

converges. Subtracting

$$\sum_{n=0}^{\infty} |a_n|,$$

which converges by assumption, we find that $\sum_{n=0}^{\infty} \mathrm{Re}(a_n)$ is the difference of two convergent series and so converges.

A similar argument works for $\mathrm{Im}(a_n)$. Finally the inequality amounts to the triangle inequality. □

Under the conditions of the hypothesis, we say that the series converges absolutely, so we can rephrase this as the statement that if a series converges absolutely, then it converges.

As an application, we come back to the geometric series, which tells us that

$$\left|\frac{1}{1-z}\right| = \left|\sum_{n=0}^{\infty} z^n\right| \le \sum_{n=0}^{\infty} |z|^n = \frac{1}{1-|z|}.$$

A consequence is that

$$\left|\frac{1}{1-z} - \sum_{k=0}^{n} z^k\right| = \left|\sum_{k=n+1}^{\infty} z^k\right| = \left|z^{n+1}\sum_{k=0}^{\infty} z^k\right| \le \frac{|z|^{n+1}}{1-|z|}$$

whenever $|z| < 1$.

2. Uniform convergence

When talking about sequences of real or complex numbers, convergence is a well-defined notion. For series, we now have two notions of convergence, the usual notion and absolute convergence. Given a sequence of functions $\{f_n\}$, we have to be more careful.

The first notion of convergence is perhaps the most obvious one: if E is some set (typically a domain in practice, but not necessarily) and $f_n : E \to \mathbb{C}$ are functions, say that the sequence $\{f_n\}$ converges pointwise if for every $z \in E$, the sequence $\{f_n(z)\}$ converges. If so, we get a new function $f : E \to \mathbb{C}$ given by

$$f(z) = \lim_{n\to\infty} f_n(z).$$

However, this notion of convergence is not totally well-behaved with respect to properties of functions. For example, it is possible to find sequences of continuous functions which converge pointwise but whose limit is not continuous. For example, if $E = [0,1]$ and $f_n(x) = x^n$, each f_n is continuous, and for any $0 \le x \le 1$ the limit $\lim_{n\to\infty} x^n$ exists; but the limiting function is

$$f(x) = \left\{ \begin{array}{ll} 0 & x < 1 \\ 1 & x = 1 \end{array} \right. ,$$

which is discontinuous at 1.

In particular, we'll want to be able to interchange limits of sequences of functions with constructions like integrals and derivatives; but if these limits don't even preserve continuity then we cannot hope to do this (e.g. the derivatives of each $f_n(x) = x^n$ are all defined, but the derivative of f is not (at $x = 1$), so one cannot hope to write it as the limit of the derivatives). So we need a stronger notion of convergence, designed specifically for functions, to replace pointwise convergence. This is given by uniform convergence.

We say that $\{f_n\}$ converges uniformly if there exists some function $f : E \to \mathbb{C}$ and a sequence of positive real numbers $\{\epsilon_n\}$ converging to 0 such that for every $z \in E$ we have

$$|f_n(z) - f(z)| < \epsilon_n.$$

In particular, the bound ϵ_n depends on n, but not on z; this is what makes this "uniform." We think of ϵ_n as measuring the maximum of the difference between f_n and f, the "worst case" of the approximation.

In the case above, for $x = 1 - \delta$ we have $|f_n(x) - f(x)| = |(1-\delta)^n - 0| = |1 - \delta + n\delta^2 - \cdots + (-1)^n \delta^n|$ which as $\delta \to 0$ approaches 1, so we cannot choose such a sequence $\{\epsilon_n\}$ approaching 0. In fact we would be forced to take $\epsilon_n \geq 1$ for all n; the worst case bound is 1.

This is supposed to explain the failure of continuity of $f(x)$. Indeed, we have the following more general theorem (whose proof is standard in real analysis and omitted here):

Theorem. *If $\{f_n\}$ is a sequence of continuous functions $E \to \mathbb{C}$ converging uniformly to a function $f : E \to \mathbb{C}$, then f is also continuous.*

We'll see next week that uniform convergence also guarantees us the other good behavior we're looking for, e.g. with respect to derivatives and integrals. This established, we'll be able to talk more meaningfully about the convergence of series of *functions* and their properties, which is the main focus of this section.