Lecture 13: Pompeiu's formula

Complex analysis, lecture 4
October 6, 2025

1. Conformal maps

Last time, we introduced the operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

for $z = x + iy$. We saw that an equivalent form of the Cauchy–Riemann equations is

$$\frac{\partial f}{\partial \overline{z}} = 0,$$

and if this holds then

$$\frac{\partial f}{\partial z} = f'(z).$$

If $f(z)$ is smooth (that is, smooth as a function of x and y, i.e. on $\mathbb{R}^2$; equivalently, all partial derivatives with respect to x and y exist[1]) then, since all partial derivatives commute with each other, we have

$$\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}.$$

In particular, if f is analytic, so $\frac{\partial f}{\partial \overline{z}} = 0$, then $0 = \frac{\partial}{\partial z}\frac{\partial f}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}\frac{\partial f}{\partial z} = \frac{\partial}{\partial \overline{z}} f'(z)$, so $f'(z)$ is also analytic. This gives a cleaner proof of one of the challenge problems from the exam.

We can write out the linear approximation formula for any smooth $f(z)$ in terms of $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$: for z close to z_0, we have

$$f(z) = f(z_0) + \frac{\partial f}{\partial z}(z_0)(z - z_0) + \frac{\partial f}{\partial \overline{z}}(z_0)(\overline{z - z_0}) + R(z - z_0)$$

where R is some complex function defined near z_0 with the property that

$$\lim_{h \to 0} \frac{R(h)}{|h|} = 0.$$

This can be proven by taking the usual multivariable expansion and rewriting in terms of our definitions here. When f is analytic, so $\frac{\partial f}{\partial \overline{z}} = 0$, this just gives the usual linear approximation formula for a differentiable function of one variable z.

We'll use this to prove a claim we made a few weeks ago:

[1] We will sometimes say "smooth" when we actually only need to assume that the real and imaginary parts are continuously differentiable, and may or may not say so.

Proposition. *Let $f : D \to \mathbb{C}$ be a smooth function on a domain D. Suppose that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are never simultaneously zero and that f is conformal. Then f is analytic on D, and $f'(z) \neq 0$ for any $z \in D$.*

This is the converse of the observation that an analytic function with nonvanishing derivative is conformal. The additional assumptions aren't actually requiring anything more: conformal functions are always taken to be smooth, and (as we discussed when we first talked about conformal functions) the derivatives can't vanish in order for the notion of being conformal to even make sense, since we can't talk about the angles between vectors of length 0.

Let $\gamma : [0, 1] \to D$ be a curve with $\gamma(0) = z_0$. To use the condition that f is conformal, we'd like to compute the tangent vector to $f \circ \gamma$ at 0 in terms of the tangent vector to γ at 0. For each $t > 0$, we have

$$f(\gamma(t)) - f(\gamma(0)) = f(\gamma(t)) - f(z_0) = \frac{\partial f}{\partial z}(z_0)(\gamma(t) - \gamma(0)) + \frac{\partial f}{\partial \overline{z}}\overline{(\gamma(t) - \gamma(0))} + R(\gamma(t) - z_0)$$

by the linear approximation formula above. Dividing by t and taking the limit as $t \to 0$, we get

$$(f \circ \gamma)'(0) = \frac{\partial f}{\partial z}(z_0)\gamma'(0) + \frac{\partial f}{\partial \overline{z}}\overline{\gamma'(0)},$$

with the remainder term getting killed off by assumption.

Suppose $\gamma(t) = z_0 + te^{i\theta}$ for some θ. Then $\gamma'(0) = e^{i\theta}$ and $\overline{\gamma'(0)} = e^{-i\theta}$, so we can rewrite this as

$$(f \circ \gamma)'(0) = e^{i\theta}\left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}}e^{-2i\theta}\right).$$

If instead we took that path $\gamma_0(t) = z_0 + t$, i.e. at $\theta = 0$, we would get the same thing but with $\theta = 0$. Since f is conformal and the angle between these paths is θ, the angle between $f \circ \gamma$ and $f \circ \gamma_0$ should also be θ, i.e.

$$(f \circ \gamma)'(0) = e^{i\theta}(f \circ \gamma_0)'(0),$$

so in other words

$$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial z}e^{-2i\theta}$$

must be independent of θ. The only way this is possible is if $\frac{\partial f}{\partial \overline{z}} = 0$ (you can see this formally by differentiating with respect to θ), so this must hold; in other words f must be analytic. The property that $f'(z) \neq 0$ everywhere then follows from the assumption on the partial derivatives of f.

2. Pompeiu's formula

When we first introduced Cauchy's formula, we described it as the complex version of half of Green's theorem. In our new language, we can write it as follows: if $\frac{\partial f}{\partial z} = 0$, then

$$\int_{\partial D} f(z)\, dz = 0.$$

We can now come back to the "full" version of Green's theorem: recall that this says

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

We have $f(z)\,dz = f(z)\,dx + if(z)\,dy$, so $P = f$ and $Q = if$; therefore the right-hand side is

$$\iint_D \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = i\iint_D \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dy = 2i\iint_D \frac{\partial f}{\partial \overline{z}}\,dx\,dy.$$

That is, if D is a bounded domain with piecewise smooth boundary and f is a smooth function on $D \cup \partial D$, then

$$\int_{\partial D} f(z)\,dz = 2i\iint_D \frac{\partial f}{\partial \overline{z}}\,dx\,dy.$$

If f is analytic, we recover Cauchy's formula.

Proceeding exactly how we derived Cauchy's formula from Cauchy's theorem using the more general formula above to incorporate a correction term, we can get the following generalization:

Theorem (Pompeiu's formula)**.** *If f is a smooth complex-valued function on $D \cup \partial D$ as above, then for any $z \in D$ we have*

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z - z_0}\,dz - \frac{1}{\pi}\iint_D \frac{\partial f}{\partial \overline{z}} \cdot \frac{1}{z - z_0}\,dx\,dy.$$

When f is analytic, the correction term is zero and we recover Cauchy's formula.

Note: for Cauchy's formula, we could differentiate under the integral sign and deduce that f was therefore infinitely (complex) differentiable. Can we do so here? No: if so, we'd find that every smooth function (in the real sense) is analytic, which is not true! The issue is that the integrand in the second term fails to be differentiable at the point $z = z_0$. This didn't come up in Cauchy's formula, where we just have the first term, since there $z \in \partial D$ rather than in D, so $z = z_0$ is impossible, but in the second term it does occur, so while the integral will still converge we can't differentiate under the integral sign.

For example, consider $f(z) = \overline{z}$ and D the open disk of radius 1 centered at the origin. For the first term, notice that for $|z| = 1$ we have $\overline{z} = \overline{e^{i\theta}} = e^{-i\theta} = \frac{1}{z}$, so this integral is the same as

$$\frac{1}{2\pi i}\int_{\partial D} \frac{1}{z(z - z_0)}\,dz.$$

There are a few ways we could compute this, e.g. via explicit parametrization. Let's instead use Cauchy's formula: just like last week, we can reduce this to the integral around two small disks centered at 0 and z_0, giving

$$\left.\frac{1}{z}\right|_{z=z_0} + \left.\frac{1}{z - z_0}\right|_{z=0} = \frac{1}{z_0} - \frac{1}{z_0} = 0.$$

Since $f(z_0)$ is nonzero in general, Cauchy's formula alone fails! We need the correction term.

Here, observe

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial \overline{z}}{\partial x} + i\frac{\partial \overline{z}}{\partial y}\right) = \frac{1}{2}\left(1 - i^2\right) = 1$$

(not unsurprisingly!) so the second term is

$$-\frac{1}{\pi}\iint_D \frac{1}{z - z_0}\, dx\, dy.$$

One can, with some care, evaluate this integral directly; Pompeiu's formula tells us much more directly that the whole thing must be equal to $\overline{z}_0$, or in other words

$$\iint_{|z|<1} \frac{1}{z - z_0}\, dx\, dy = -\pi\overline{z}_0,$$

which is perhaps not a formula one would otherwise guess.