Lecture 12: applications of Cauchy's formula to analyticity

Complex analysis, lecture 4

October 3, 2025

Last time: the Cauchy estimate

We use Cauchy's formula for higher derivatives to show that if f is analytic on $\{z : |z - z_0| \leq r\}$ and $|f(z)| \leq M$ for $|z - z_0| = r$, then

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} M.$$

This has the following very nice application.

Liouville's theorem

- Say a function is entire if it is analytic on the whole complex plane.
- Recall also that a function f is bounded if there exists a real number M such that for any z in the domain of f, $|f(z)| \leq M$.

Theorem (Liouville's theorem)

Any bounded entire function is constant.

- e.g. $\sin z$ and $\cos z$ are analytic everywhere as complex functions, hence not bounded!
- in particular the familiar bounds $|\sin z| \leq 1$, $|\cos z| \leq 1$ over $\mathbb{R}$ are not true over $\mathbb{C}$.

Fundamental theorem of algebra

Another proof of the fundamental theorem of algebra: if $P(z)$ has no roots in $\mathbb{C}$, then $\frac{1}{P(z)}$ is entire. Either

- $\deg P = 0$, so $P(z)$ and $\frac{1}{P(z)}$ are constant;
- or $\deg P \geq 1$, so $\lim_{|z|\to\infty} |P(z)| = \infty$, so $\lim_{|z|\to\infty} \frac{1}{|P(z)|} = 0$, so $\frac{1}{|P(z)|}$ is bounded.

Either way $\frac{1}{P(z)}$ is bounded, so by Liouville's theorem it is constant. Hence if P has no zeros in $\mathbb{C}$ it is constant, i.e. any non-constant polynomial has a zero in $\mathbb{C}$.

Proof of Liouville's theorem

If f is bounded ($|f(z)| \leq M$ for all z) and entire, for any $r > |z|$ by the Cauchy estimates for $n = 1$

$$|f'(z)| \leq \frac{1}{r} \cdot M.$$

Taking $r \to \infty$, it follows that $f'(z) = 0$ for all z, so f is constant.

Morera's theorem

Recall: if $f(z)$ is analytic on a domain D, then $f(z)\,dz$ is closed on D. (So $\int_{\partial D} f(z)\,dz = \int_D 0\,dz = 0$, Cauchy's theorem.)

Morera's theorem: loosely speaking, the converse holds. More precisely:

Theorem (Morera's theorem, version 1)

Let f be a continuous function on a domain D. If for any sub-domain $D' \subset D$ we have

$$\int_{\partial D'} f(z)\,dz = 0,$$

then f is analytic on D.

Morera's theorem

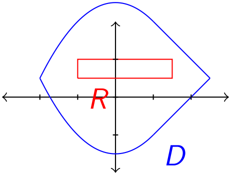
Actually, we just need to check the case when D' is a rectangle of a certain type:

Theorem (Morera's theorem, version 2)

Let f be a continuous function on a domain D. If for any rectangle $R \subset D$ with sides parallel to the coordinate axes we have

$$\int_{\partial R} f(z)\, dz = 0,$$

then f is analytic on D.



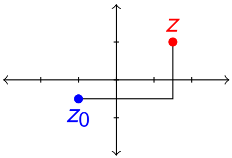
Proof of Morera's theorem

It suffices to prove version 2. Actually we can also assume D is a disk, by picking a small disk around any point in D and using Morera's theorem for disks to show that f is analytic there, so assume D is a disk centered at z_0.

In this case exact = closed, so we expect $f = F'$ for

$$F(z) = \int_{z_0}^{z} f(w)\, dw$$

where the path is horizontal and then vertical.

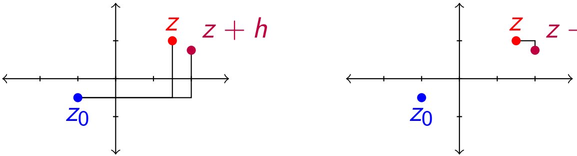


Proof of Morera's theorem

Fix a small complex number h with $|h| < \epsilon$. Then

$$F(z+h) - F(z) = \int_z^{z+h} f(w)\, dw$$

where the path is from z to z_0 (reverse of the above) to $z+h$ (as above) or equivalently moving the line:



Here we are using the hypothesis that the integral over the boundary of a rectangle is 0.

Proof of Morera's theorem

Now

$$\int_z^{z+h} f(w)\,dw = \int_z^{z+h} f(z) + (f(w) - f(z))\,dw$$
$$= hf(z) + \int_z^{z+h} (f(w) - f(z))\,dw.$$

Since f is continuous, for any $\delta > 0$ we have $|f(w) - f(z)| < \delta$ for ϵ small enough, and the length of the path is $\leq 2|h|$, so by the ML bound:

$$|F(z+h) - F(z) - hf(z)| < 2\delta|h|.$$

Proof of Morera's theorem

Divide by h:

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right|<2\delta$$

so since we can take δ as small as we like as $h \to 0$ this is 0.

Therefore F is analytic, so so is $F' = f$ (by Cauchy's formula for derivatives!).

Goursat's theorem

An analytic function f on a domain D:

1. has to be complex differentiable on D,
2. and f' has to be continuous on D.

We claimed that (2) is redundant. We can now justify this:

Theorem (Goursat's theorem)

If $f : D \to \mathbb{C}$ is a function such that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for every $z_0 \in D$, then f is analytic (i.e. f' is also continuous).

Proof of Goursat's theorem

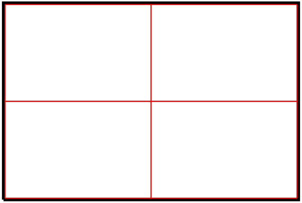
Note that f is automatically continuous, or else the limit won't exist, so we can try to use Morera's theorem.

Let $R \subset D$ be a rectangle. We want to show

$$\int_{\partial R} f(z)\, dz = 0.$$

Proof of Goursat's theorem

Divide R into four equal rectangles R_1, R_2, R_3, R_4:



Note

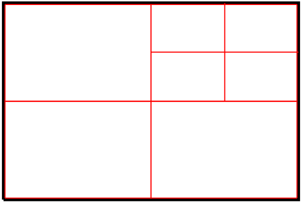
$$\int_{\partial R} f(z)\,dz = \int_{\partial R_1} f(z)\,dz + \int_{\partial R_2} f(z)\,dz + \int_{\partial R_3} f(z)\,dz + \int_{\partial R_4} f(z)\,dz$$

so one of the R_i satisfies

$$\left| \int_{\partial R_i} f(z)\,dz \right| \geq \frac{1}{4} \left| \int_{\partial R} f(z)\,dz \right| .$$

Proof of Goursat's theorem

Relabel this R_i to R^1 and repeat the process:

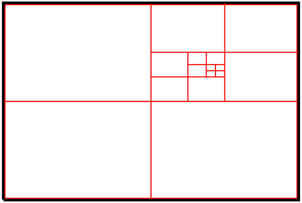


One of the inner rectangles (call it R^2) must satisfy

$$\left| \int_{\partial R^2} f(z)\, dz \right| \geq \frac{1}{4} \left| \int_{\partial R^1} f(z)\, dz \right| \geq \frac{1}{4^2} \left| \int_{\partial R} f(z)\, dz \right|.$$

Proof of Goursat's theorem

Continue indefinitely:



We get a sequence of $R \supset R^1 \supset R^2 \supset \cdots$ with

$$\left| \int_{\partial R^n} f(z)\, dz \right| \geq \frac{1}{4^n} \left| \int_{\partial R} f(z)\, dz \right|.$$

Proof of Goursat's theorem

Since the diameters of the R^n are decreasing to 0 and they all include each other, they are converging towards some point z_0.

Since f is differentiable at z_0, for $z \in R^n$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon_n$$

for $\epsilon_n \to 0$ as $n \to \infty$.

Let L be the length of ∂R, so the length of ∂R^n is $L/2^n$. Then

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon_n |z - z_0| \leq \epsilon_n L/2^n.$$

Proof of Goursat's theorem

Note $f(z_0) + f'(z_0)(z - z_0)$ is analytic everywhere, so by Cauchy's theorem

$$\int_{\partial R^n} (f(z_0) + f'(z_0)(z - z_0))\, dz = 0.$$

Therefore by the ML bound

$$\begin{aligned} \left| \int_{\partial R^n} f(z)\, dz \right| &= \left| \int_{\partial R^n} (f(z) - f(z_0) - f'(z_0)(z - z_0))\, dz \right| \\ &\leq \frac{L}{2^n} \cdot \epsilon_n \cdot \frac{L}{2^n} \\ &= \frac{L^2}{4^n} \cdot \epsilon_n. \end{aligned}$$

Proof of Goursat's theorem

Therefore:

$$\left| \int_{\partial R} f(z)\, dz \right| \leq 4^n \left| \int_{\partial R^n} f(z)\, dz \right| \leq L^2 \epsilon_n$$

for all n.

Since $\epsilon_n \to 0$ as $n \to 0$, the integral must be 0. So by Morera's theorem f is analytic.

The operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$

When working with the Cauchy–Riemann equations, we needed to talk about the real and imaginary parts u and v of f, and the x and y partial derivatives. We would rather have something more complex-looking.

Define operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial (iy)}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial (iy)}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$

If f is analytic,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial (iy)} = -i\frac{\partial f}{\partial y}$$

using the Cauchy–Riemann equations. Averaging:

$$f'(z) = \frac{1}{2}(f'(z) + f'(z)) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial z}.$$

Complex form of Cauchy–Riemann

On the other hand:

$$0 = \frac{1}{2}(f'(z) - f'(z)) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial \bar{z}}.$$

So f analytic $\implies \frac{\partial f}{\partial \bar{z}} = 0$.

More generally if $f = u + iv$,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right).$$

Complex form of Cauchy–Riemann

Taking real and imaginary parts: $\frac{\partial f}{\partial \bar{z}} = 0$ if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

the Cauchy–Riemann equations.

So an equivalent form is

$$\frac{\partial f}{\partial \bar{z}} = 0,$$

sometimes called the complex form of the Cauchy–Riemann equations.

Properties

The operators $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$ act like partial derivative operators:

$$\frac{\partial}{\partial z}(af+bg) = a\frac{\partial f}{\partial z} + b\frac{\partial g}{\partial z}, \qquad \frac{\partial}{\partial \bar{z}}(af+bg) = a\frac{\partial f}{\partial \bar{z}} + b\frac{\partial g}{\partial \bar{z}}$$

for constants a, b, and

$$\frac{\partial}{\partial z}(fg) = f\frac{\partial g}{\partial z} + \frac{\partial f}{\partial z}g, \qquad \frac{\partial}{\partial \bar{z}}(fg) = f\frac{\partial g}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{z}}g.$$

Properties

They also satisfy

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}}, \qquad \overline{\frac{\partial \bar{f}}{\partial z}} = \frac{\partial f}{\partial \bar{z}}.$$

So e.g. if f and $\bar{f}$ are both analytic, then

$$0 = \frac{\partial \bar{f}}{\partial \bar{z}} = \overline{\frac{\partial f}{\partial z}}$$

and so f is constant.

Next time we'll make some more serious use of these operators.