Lecture 11: using Cauchy's formula

Complex analysis, lecture 4
October 1, 2025

Last time, we studied complex path integrals. Most directly, we saw how to compute them by parametrizing the path. More abstractly, we observed that if $f(z)$ is analytic on a domain D, then $f(z)\,dz$ is closed; so if D is star-shaped, then $f(z)\,dz$ is exact, and so we can find a primitive F for f, i.e. an analytic function F on D such that $F' = f$, and then we can evaluate path integrals on D using the fundamental theorem of calculus.

The higher-level version of this is Cauchy's theorem: if D is a bounded domain with boundary ∂D piecewise smooth and f is analytic on D, extending smoothly to the boundary ∂D, then

$$\int_{\partial D} f(z)\,dz = 0.$$

As a corollary, we have Cauchy's theorem: for any $z \in D$,

$$f(z) = \frac{1}{2\pi i}\int_D \frac{f(w)}{w - z}\,dz.$$

Today, we want to give some first applications of Cauchy's theorem and formula. The most straightforward application is the computation of integrals: if f is analytic on a region D, then Cauchy's theorem tells us that its integral over the boundary of D is zero. Similarly, Cauchy's formula tells us that the integral over a domain D of anything of the form $\frac{f(z)}{z-z_0}$ is just $2\pi i f(z)$.

For example, consider the disk $D = \{z : |z| < 3\}$, so ∂D is the circle γ of radius 3. Then

$$\int_\gamma \frac{z^2}{z+2}\,dz = 2\pi i \cdot (-2)^2 = 8\pi i.$$

An interesting simpler example is, for γ any simple closed curve bounding a domain containing the origin,

$$\int_\gamma \frac{1}{z}\,dz = 2\pi i.$$

When γ is a circle, this matches our calculation last time.

In fact, we can push this method a little further: recall we had the more general formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{n+1}}\,dw,$$

so we can evaluate anything of this form as well. For example, for γ the circle of radius 3 as above,

$$\int_\gamma \frac{e^z}{(z+2)^3}\,dz$$

comes from the $n = 2$ case, with $f(z) = e^z$. Since $f^{(2)}(z) = e^z$, this gives

$$\int_\gamma \frac{e^z}{(z+2)^3}\,dz = \frac{2\pi i}{2!}e^{-2} = \frac{\pi i}{e^2}.$$

This would otherwise be a very difficult integral to compute!

For a slightly harder version, consider an integral like

$$\int_\gamma \frac{e^z}{z^2(z+2)}\,dz.$$

This is not of the form we can evaluate using Cauchy's formula, even the more general version: we would need to use either $f(z) = \frac{e^z}{z^2}$ or $g(z) = \frac{e^z}{z+2}$, neither of which is analytic on the whole disk.

However, we can use the following strategy. Let D_0 be a disk of some very small radius r centered at 0, and D_{-2} another disk of radius r centered at -2 (with r small enough that they don't intersect). On $D' = D \setminus D_0 \setminus D_{-2}$, the function $h(z) = \frac{e^z}{z^2(z+2)}$ is analytic, so

$$\int_{\partial D'} h(z)\,dz = 0.$$

Now, we can write $\partial D'$ (keeping in mind orientations) as γ, in the positive direction, together with two circles of radii r centered at 0 and -2 in the negative direction, which we call γ_0 and γ_{-2} respectively. Therefore

$$0 = \int_{\partial D'} h(z) = \int_\gamma h(z)\,dz - \int_{\gamma_0} h(z)\,dz - \int_{\gamma_{-2}} h(z)\,dz,$$

i.e.

$$\int_\gamma h(z)\,dz = \int_{\gamma_0} h(z)\,dz + \int_{\gamma_{-2}} h(z)\,dz.$$

For the first term, note that on the disk D_0 of sufficiently small radius r near 0, $g(z) = \frac{e^z}{z+2}$ is analytic, and so we can use Cauchy's formula (for derivatives):

$$\int_{\gamma_0} \frac{e^z}{z^2(z+2)}\,dz = \frac{2\pi i}{1!}g'(0) = \frac{\pi i}{2}$$

(evaluating the derivative is left as an exercise for the reader). For the second term, similarly $f(z) = \frac{e^z}{z^2}$ is analytic on D_{-2} for r sufficiently small, so

$$\int_{\gamma_{-2}} \frac{e^z}{z^2(z+2)}\,dz = 2\pi i f(-2) = \frac{\pi i}{2e^2}.$$

Therefore the total integral is

$$\int_\gamma \frac{e^z}{z^2(z+2)}\,dz = \frac{\pi i}{2}(1 + e^{-2}).$$

We can also use Cauchy's formula to bound the derivatives of f. For a fixed point z_0, choose r small enough that f is defined on the closed disk $\{z : |z| \leq r\}$. By Cauchy's formula (for derivatives), we have

$$\begin{aligned} f^{(n)}(z_0) &= \frac{n!}{2\pi i} \int_{|z|=r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \\ &= \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{r^{n+1} e^{i\theta(n+1)}} rie^{i\theta}\, d\theta \\ &= \frac{n!}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta}\, dz. \end{aligned}$$

If $|f(z)| \leq M$ for $|z - z_0| = r$, the ML bound gives

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi r^n} \cdot 2\pi M = \frac{n!}{r^n} M.$$

This is the Cauchy estimate for higher derivatives.

As a corollary, we deduce the following extremely nice result. We say a function is entire if it is analytic on the whole complex plane. Recall also that a function f is bounded if there exists a real number M such that for any z in the domain of f, $|f(z)| \leq M$.

Theorem (Liouville's theorem)**.** *Any bounded entire function is constant.*

This is very surprising: in the real case, we have plenty of very nicely behaved non-constant analytic functions which are bounded, e.g. $\sin(x)$ and $\cos(x)$. We have seen that these functions have nice extensions to the complex plane, and are analytic everywhere, hence entire; so Liouville's theorem tells us that, somewhat counter-intuitively, $\sin z$ and $\cos z$ are not bounded in the complex setting (since they are entire and not constant). In particular the bounds $|\sin(z)| \leq 1$, $|\cos(z)| \leq 1$ are false in general for z complex.

Given the Cauchy estimate, the proof is now straightforward. If f is bounded and entire, we fix some M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. The Cauchy estimate tells us that $|f^{(n)}(z)| \leq \frac{n!}{r^n} M$ for any $r > |z|$. Since we can take r arbitrarily large (since f is entire), we can take the right-hand side arbitrarily close to zero, so in fact we must have $|f^{(n)}(z)| = 0$ for all n and all z. In fact the case $n = 1$ is enough: this shows that $f'(z) = 0$ for all z, hence f is constant.

Liouville's theorem gives yet another proof of the fundamental theorem of algebra: suppose that $P(z) = a_n z^n + \cdots + a_1 z + a_0$ is a complex polynomial with no roots in $\mathbb{C}$. Then $\frac{1}{P(z)}$ is an entire function. If $n \geq 1$, then $|P(z)| \to \infty$ as $|z| \to \infty$, so $\left|\frac{1}{P(z)}\right| \to 0$ as $|z| \to \infty$; therefore $\frac{1}{P}$ is bounded. (In fact, if $n = 0$ then P is constant so again bounded.) Therefore by Liouville's theorem $\frac{1}{P}$ must be constant, i.e. P is constant, so the only polynomials with no complex zeros are the constant ones, i.e. the fundamental theorem of algebra holds.