Lecture 10: complex integrals

Complex analysis, lecture 4
September 29, 2025

Last week, we reviewed path integrals in the real plane and some of their applications. This week, we'll specialize to the complex plane and see what things look like here.

We start off by writing $dz = dx + i\,dy$ for $z = x + iy$. Thus for $h : D \to \mathbb{C}$ a complex function on a domain D and γ a curve in D, if $h = u + iv$ we have

$$\begin{aligned}\int_\gamma h(z)\,dz &= \int_\gamma h(z)\,dx + ih(z)\,dy \\ &= \int_\gamma (u+iv)\,dx + (-v+iu)\,dy.\end{aligned}$$

In particular, the differential $\omega = h(z)\,dz = (u+iv)\,dx + (-v+iu)\,dy$ is closed if and only if

$$\frac{\partial}{\partial y}(u+iv) = \frac{\partial}{\partial x}(-v+iu),$$

taking the real and imaginary parts of which give

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x},$$

which are exactly the Cauchy–Riemann equations! That is: $h(z)\,dz$ is closed on D if and only if h is analytic on D.

Before proceeding further, let's compute an example. Let γ be the circle around a point z_0 (in the positive direction) of radius R, which in the complex setting we can parametrize by $z = \gamma(\theta) = z_0 + Re^{i\theta}$ for $0 \le \theta < 2\pi$. Then $dz = iRe^{i\theta}\,d\theta$. For $f(z) = (z-z_0)^n$, the integral is

$$\int_0^{2\pi} (Re^{i\theta})^n iRe^{i\theta}\,d\theta = iR^{n+1}\int_0^{2\pi} e^{i\theta(n+1)}\,d\theta.$$

For $n+1 \neq 0$, this integral is zero, by symmetry or by expressing in terms of trigonometric functions; when $n+1 = 0$, i.e. when $n = -1$, it is just the integral of the constant function 1, multiplied by $iR^{n+1} = i$, so

$$2\pi i.$$

That is,

$$\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i, \qquad \int_\gamma (z-z_0)^n\,dz = 0$$

for $n \neq -1$. This will actually be an important calculation later.

Note that although this is a path integral, in many respects we could now treat it as similar to a usual one-variable integral, just now with complex parameter and output. This is an advantage of working over $\mathbb{C}$.

Integrals over $\mathbb{C}$ satisfy the following useful bound: if γ is a curve in $\mathbb{C}$ and h is a continuous complex function defined on γ, then

$$\left| \int_\gamma h(z)\, dz \right| \leq \int_\gamma |h(z)|\, |dz|.$$

(This is essentially the triangle inequality.) If γ has length $L = \int_\gamma |dz| = \int_\gamma |dx + i\, dy| = \int_\gamma \sqrt{x'(t)^2 + y'(t)^2}\, dt$ and for every $z \in \gamma$ we have $|h(z)| \leq M$, then

$$\left| \int_\gamma h(z)\, dz \right| \leq \int_\gamma |h(z)|\, |dz| \leq \int_\gamma M\, |dz| = ML.$$

This is sometimes simply called the ML bound, and is surprisingly useful. We will sometimes invoke both bounds above without explicitly referring to them, so they're good to know implicitly.

For the example above, on the circle γ of radius R around z_0, we have $|f(z)| = |(z - z_0)^n| = R^n$ and γ has length $2\pi R$, so the estimate bounds the integral by $2\pi R^{n+1}$. For $n \neq -1$, the actual value for the integral is zero, so this is not a very good bound. For $n = -1$ however the integral is $2\pi i$, with absolute value 2π, while the bound is 2π, so it is actually sharp.

Much like in the real case, we can ask if a version of the fundamental theorem of calculus holds in our setting. The answer is yes: if $f(z) = F'(z)$ for an analytic function F on D, then $F'(z) = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial (iy)} = \frac{1}{i}\frac{\partial F}{\partial y}$ (note the factor of i!), and so

$$F(B) - F(A) = \int_A^B dF = \int_A^B \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy = \int_A^B F'(z)(dx + i\, dy) = \int_A^B F'(z)\, dz.$$

In particular this is independent of the path taken in the complex plane.

However, we have to be careful. If $A = B$, this would predict that the integral is always zero. But we saw above an example where this is not true: $f(z) = \frac{1}{z - z_0}$ around a circle centered at z_0. Indeed, although we can find an antiderivative of f here, given by $\log(z - z_0) + C$ for some constant C, it is not defined on the interior of the disk, in particular at z_0, so there is no analytic primitive on D. Therefore the integral is not necessarily path-independent; indeed taking the constant path from some point on the circle to itself would give 0, different from the circle path.

However, we've already seen how to fix this. Just as in the real case, a differential is exact if it comes from an analytic function; it is closed if it satisfies the corresponding differential equation, and we saw above that if $h(z)$ is analytic, then $h(z)\, dz$ is closed. So in the case where closed and exact are the same, it follows that $h(z)\, dz$ is exact. In particular, if D is a star-shaped domain, then

$$\int_\gamma h(z)\, dz$$

is independent of the path γ in D so long as $h(z)$ is analytic on D.

We now have two methods to compute complex integrals: by direct parametrization, and (if we're in a star-shaped domain, or otherwise know our differential to be exact) by the fundamental theorem of calculus. In the real case, we had another method: Green's theorem. This told us two things: first, that the integral around the boundary of a domain of a closed differential vanishes; and second, for a non-closed differential, it gave us an exact formula, given by an integral over the interior of the domain.

What might a complex analogue look like? Let's start with the first part. We said that if $h(z)$ is analytic on a (bounded) domain D (with piecewise smooth boundary), then $h(z)\,dz$ is closed; so by Green's theorem,

$$\int_{\partial D} h(z)\,dz = 0.$$

Despite the simplicity of its proof, this is a very important theorem, called Cauchy's integral theorem.

Consider the region $D = \{r < |z| < R\}$ for $0 < r < R$ real numbers. This is an annulus around the origin, i.e. a disk with a smaller disk cut out. Its boundary is a pair of circles: one of radius R, in the positive direction, and one of radius r, in the *negative* direction. This has to do with the notion of orientation, which we have only briefly touched on; for the boundary of any domain D, we want to think of it as going in the direction to have the interior of D on its left. So for a disk, the boundary is the circle in the positive direction; but for the annulus, the inner boundary is in the negative direction.

By Cauchy's theorem, for any h analytic on D we have

$$0 = \int_{\partial d} h(z)\,dz = \oint_{|z|=R} h(z)\,dz - \oint_{|z|=r} h(z)\,dz.$$

Here $\oint$ denotes an integral around a closed loop; we will sometimes use this as a notational reminder. In particular, it follows that

$$\oint_{|z|=R} h(z)\,dz = \oint_{|z|=r} h(z)\,dz,$$

i.e. $\oint_{|z|=r} h(z)\,dz$ is independent of r (so long as h is analytic on the annulus containing the circle of radius r). This makes sense with our discussion of deforming the paths along which we integrate. For $h(z) = z^n$, it also agrees with our calculations above.

Cauchy's integral theorem has many applications, but one of the first and most critical is also due to Cauchy and is called Cauchy's integral formula (somewhat confusingly).

Theorem (Cauchy's integral formula)**.** *Let D be a bounded domain with piecewise smooth boundary. If $h(z)$ is analytic on D and extends smoothly to the boundary of D, then*

$$h(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{h(w)}{w-z}\,dw$$

for any $z \in D$.

This is quite a remarkable formula. The right-hand side only involves evaluating h at points on the boundary of the domain, but we are claiming that somehow this can tell us the value of the function at every point on the interior! Further, this is true for any D, so if the function is defined on some larger domain we can take any domain inside that containing z and use this boundary; so this is a strong deformation statement as well.

The proof is as follows. Let U be a small disk of radius r centered at z, contained in D. We can consider the domain $D \setminus U$, with boundary $\partial D \cup \gamma$ where γ is the circle of radius r around z in the negative direction. Then $\frac{h(w)}{z-w}$ as a function of w is analytic on $D \setminus U$, since there $w \neq z$; so by Cauchy's theorem,

$$\int_{\partial D} \frac{h(w)}{z-w}\, dw - \int_{\gamma} \frac{h(w)}{z-w}\, dw = 0.$$

So it suffices to study the second term, which is then equal to the first.

On the circle, we can write $w = z + re^{i\theta}$, with $dw = ire^{i\theta}\, d\theta$, so the integral becomes

$$\int_0^{2\pi} \frac{h(z + re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\, d\theta = 2\pi i A(r)$$

where $A(r)$ is the mean value on the circle of radius r as last time. Since h is analytic, it is harmonic, so it satisfies the mean value property, hence $A(r) = h(z)$. Therefore the second term is $2\pi i h(z)$, i.e.

$$\frac{1}{2\pi i} \int_{\partial D} \frac{h(w)}{z-w}\, dw = h(z).$$

Differentiating, we can even get a formula for every derivative at z:

$$h^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}}\, dw$$

for every $n \geq 0$ and $z \in D$. This is sometimes useful for things like computing Taylor series. More abstractly, we have justified one of our big claims about analytic functions: this shows that, even though we only assumed f is continuously differentiable on D, it is actually infinitely differentiable at every point in D! This is even more valuable than the explicit formula. Note that unlike the exam problem, we didn't have to make any assumptions about the differentiability of its real and imaginary parts; we get all this for free from Cauchy's formula.

Eventually, we'll also want to show that holomorphic functions are "analytic" in the sense of being equal to their Taylor series, but as we haven't really talked about Taylor series yet we'll put this off.