Homework 9

Complex analysis, lecture 4

Due November 24, 2025 by 11:59 PM

As usual, you may use any resources to solve these problems except where stated otherwise, with the exception of computational software/generative AI and posting these problems anywhere to be answered by others. Collaboration is encouraged, but everyone should write their own solutions. Write the names of any collaborators or sources used at the top of your homework. If you did not use any sources, write "sources used: none."

If you find any errors in either the homework or the lecture notes, please let me know, even if you are unsure whether it is an error or not.

As on most math problems, the mathematics is the issue, not the answer: whether you have a correct method is more important then whether you get to the correct number at the end, so include your method!

You do not have to simplify your answers completely (so for example $\frac{2}{2}$ is fine), but you do need to do all the computations (so for example if the problem is "find the largest value of $f(x)$," the answer "$f(3)$" is incomplete; you would also need to evaluate f at 3).

Problem 1. Let $f(z) = \frac{z^9+2z^5-2z^4+z+3}{2z-1}$. The goal of this problem is to count the number of zeros of f with real part greater than 0.

(a) Let $D = \{z : \mathrm{Re}(z) > 0, |z| < R\}$ for some large real number R, so that its boundary is given by the arc of radius R from angle $-\pi/2$ to $\pi/2$ together with the line segment connecting Ri to $-Ri$. Show that (for R sufficiently large) f has no zeros or poles on ∂D.

(b) Write $g(z) = z^9 + 2z^5 - 2z^4 + z + 3$ and $h(z) = 2z - 1$, so that $f(z) = \frac{g(z)}{h(z)}$. Compute the increase in argument of $h(z)$ as z moves along the arc. Recall that you can make an approximation for R sufficiently large.

(c) Compute the increase of argument of $h(z)$ as z moves along the line segment, from Ri to $-Ri$.

(d) Compute the increase in argument of $g(z)$ as z moves along the arc.

(e) Compute the increase of argument of $g(z)$ as z moves along the line segment.

(f) Putting everything together and using $\arg f = \arg g - \arg h$, compute the increase in argument of f along ∂D. Using the argument principle and by studying the poles of f directly, find the number of zeros of f (with multiplicity) inside D, and by taking the limit as $R \to \infty$ the number of zeros in the right half-plane.

The above problem is directed towards Objective 11 (the argument principle).

Problem 2. Using Rouché's theorem, find the number of solutions to $z^5 + 3z^2 + 1 = 0$, counting multiplicity, with $|z| < 1$.

The above problem is directed towards Objective 11 (the argument principle).

Problem 3. Show that any conformal self-map of the punctured unit disk $\{0 < |z| < 1\}$ is a rotation $z \mapsto e^{i\theta} \cdot z$ for some $0 \leq \theta < 2\pi$.

The above problem is directed towards Objective 12 (the Schwarz lemma).

Problem 4. Let $\Omega = \{z : 0 < \text{Re}(z) < 2\}$ be a vertical strip, and let $f : \Omega \to D = \{z : |z| < 1\}$ be a holomorphic map with $f(1) = 0$. Write down a conformal map $g : D \to \Omega$ with $g(0) = 1$. Show that $|f'(1)| \leq \frac{1}{|g'(0)|}$.

The above problem is directed towards Objectives 12 and 14 (the Schwarz lemma and the Riemann mapping theorem).

Problem 5. Let $\Omega \subset \mathbb{C}$ be a simply connected domain which is not all of $\mathbb{C}$, and let $f, g : D = \{|z| < 1\} \to \Omega$ be conformal mappings such that $f(0) = g(0) = z_0$. Show that there exists some θ such that $f(z) = g(e^{i\theta} z)$, i.e. the conformal map from D which is guaranteed to exist by the Riemann mapping theorem is *almost* unique.

The above problem is directed towards Objective 14 (the Riemann mapping theorem).

Survey. Estimate the amount of time you spent on each problem to the nearest half hour.

	Time Spent
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.