Homework 6

Complex analysis, lecture 4

Due October 27, 2025 by 11:59 PM

As usual, you may use any resources to solve these problems except where stated otherwise, with the exception of computational software/generative AI and posting these problems anywhere to be answered by others. Collaboration is encouraged, but everyone should write their own solutions. Write the names of any collaborators or sources used at the top of your homework. If you did not use any sources, write "sources used: none."

If you find any errors in either the homework or the lecture notes, please let me know, even if you are unsure whether it is an error or not.

As on most math problems, the mathematics is the issue, not the answer: whether you have a correct method is more important then whether you get to the correct number at the end, so include your method!

You do not have to simplify your answers completely (so for example $\frac{2}{2}$ is fine), but you do need to do all the computations (so for example if the problem is "find the largest value of $f(x)$," the answer "$f(3)$" is incomplete; you would also need to evaluate f at 3).

Problem 1. For each of the following functions, find the power series representation centered at the given point z_0, and determine its radius of convergence.

(a) $f(z) = \frac{1}{z-1}$ at $z_0 = i$;

(b) $f(z) = \operatorname{Log} z$ at $z_0 = 2$. (Note that $\operatorname{Log} z$ is analytic in a sufficiently small disk centered at $z_0 = 2$, and since $e^{\operatorname{Log} z} = z$ for all z, differentiating we find $\left(\frac{d}{dz} \operatorname{Log} z\right) \cdot e^{\operatorname{Log} z} = 1$ and so $\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}$.)

The above problem is directed towards Objective 8 (power series).

Problem 2. Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

converges on an open disk of radius R centered at the origin, and on this disk satisfies the differential equation

$$f'(z) = zf(z)$$

for all $|z| < R$. Show that the a_n must satisfy the recurrence relation $na_n = a_{n-2}$ for $n \geq 2$ and $a_1 = 0$. Conclude that

$$f(z) = a_0 \sum_{n=0}^{\infty} \frac{1}{n! 2^n} z^{2n} = a_0 e^{z^2/2}.$$

The above problem is directed towards Objective 8 (power series).

Problem 3. Suppose that f is analytic at infinity, with power series at infinity given by

$$f(1/z) = \sum_{n=0}^{\infty} a_n z^n.$$

Write $f(\infty) = a_0$ and $f'(\infty) = a_1$. Show that

$$f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty)).$$

The above problem is directed towards Objective 8 (power series).

Survey. Estimate the amount of time you spent on each problem to the nearest half hour.

	Time Spent
Problem 1	
Problem 2	
Problem 3	

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.