Homework 4

Complex analysis, lecture 4

Due October 6, 2025 by 11:59 PM

As usual, you may use any resources to solve these problems except where stated otherwise, with the exception of computational software/generative AI and posting these problems anywhere to be answered by others. Collaboration is encouraged, but everyone should write their own solutions. Write the names of any collaborators or sources used at the top of your homework. If you did not use any sources, write "sources used: none."

If you find any errors in either the homework or the lecture notes, please let me know, even if you are unsure whether it is an error or not.

As on most math problems, the mathematics is the issue, not the answer: whether you have a correct method is more important then whether you get to the correct number at the end, so include your method!

You do not have to simplify your answers completely (so for example $\frac{2}{2}$ is fine), but you do need to do all the computations (so for example if the problem is "find the largest value of $f(x)$," the answer "$f(3)$" is incomplete; you would also need to evaluate f at 3).

Problem 1. Let $f : \mathbb{C} \to \mathbb{C}$ be the function given by $f(x + iy) = \cos x + i \sin y$. Let γ be a path in $\mathbb{C}$ from $\gamma(0) = z_0$ to $\gamma(1) = z_1$. Is the integral $\int_\gamma f(z)\,dz$ independent of the choice of path γ from z_0 to z_1?

The above problem is directed towards Objective 5 (complex integration).

Problem 2. Let D be the triangle $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1, 0 < \mathrm{Im}(z) < 1 - \mathrm{Re(z)}\}$, and let γ be its boundary (in the counterclockwise direction as usual).

(a) Compute $\int_\gamma \mathrm{Re(z)}\,dz$.

(b) Find the ML bound for the above integral. Is it sharp in this case?

The above problem is directed towards Objective 5 (complex integration).

Problem 3. Suppose $P(z) = a_n z^n + \cdots + a_1 z + a_0$ is a non-constant polynomial (so $n \geq 1$, and $|P(z)| \to \infty$ as $|z| \to \infty$) with complex coefficients a_i. Let $Q(z) = \frac{1}{z}(P(z) - P(0)) = a_n z^{n-1} + \cdots + a_2 z + a_1$. Then $P(z) = a_0 + zQ(z)$, so

$$\frac{a_0 + zQ(z)}{zP(z)} = \frac{P(z)}{zP(z)} = \frac{1}{z}.$$

Integrating around a circle of sufficiently large radius R, use Cauchy's theorem and the ML bound to show that $P(z) = 0$ must have some solution with $|z| < R$ for R sufficiently large.

This gives another proof of the fundamental theorem of algebra!

The above problem is directed towards Objective 6 (Cauchy's integral theorem and formula).

Problem 4. Evaluate the integral

$$\oint_{|z|=r} \frac{\sin(z)}{z}\, dz.$$

The above problem is directed towards Objective 6 (Cauchy's integral theorem and formula).

Problem 5. Let $u : D \to \mathbb{R}$ be a harmonic function on a domain $D \subset \mathbb{R}^2$, and fix a point $z_0 \in D$ and a disk of radius r centered at z_0 and contained in D. Give a new proof of the mean value property for u,

$$\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta = u(z_0),$$

using Cauchy's integral formula.

The above problem is directed towards Objective 6 (Cauchy's integral theorem and formula).

Survey. Estimate the amount of time you spent on each problem to the nearest half hour.

	Time Spent
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.