Homework 3

Complex analysis, lecture 4

Due September 29, 2025 by 11:59 PM

As usual, you may use any resources to solve these problems except where stated otherwise, with the exception of computational software/generative AI and posting these problems anywhere to be answered by others. Collaboration is encouraged, but everyone should write their own solutions. Write the names of any collaborators or sources used at the top of your homework. If you did not use any sources, write "sources used: none."

If you find any errors in either the homework or the lecture notes, please let me know, even if you are unsure whether it is an error or not.

As on most math problems, the mathematics is the issue, not the answer: whether you have a correct method is more important then whether you get to the correct number at the end, so include your method!

You do not have to simplify your answers completely (so for example $\frac{2}{2}$ is fine), but you do need to do all the computations (so for example if the problem is "find the largest value of $f(x)$," the answer "$f(3)$" is incomplete; you would also need to evaluate f at 3).

Problem 1. Let $D, D' \subseteq \mathbb{C}$ be domains, and let $f : D \to D'$ be an analytic conformal mapping, so f is bijective with inverse f^{-1}.

(a) Show that f^{-1} is also a conformal mapping.

(b) If $u : D' \to \mathbb{R}$ is a harmonic function (viewing $D \subset \mathbb{C} \simeq \mathbb{R}^2$ as a subset of the plane), show that $u \circ f : D \to \mathbb{R}$ is again harmonic. (Hint: use Laplace's equation for harmonic functions, the fact that analytic functions are harmonic, and the Cauchy–Riemann equations.) Combining this with part (a), explain how this gives a bijection between harmonic functions on D and on D'.

The above problem is directed towards Objective 4 (harmonic functions and conformal mappings).

Problem 2. Let $p(z)$ be a nonconstant polynomial with complex coefficients. Use the harmonic maximum principle for $h(z) = \frac{1}{p(z)}$ to show that the equation $p(z) = 0$ has a solution in $\mathbb{C}$.

To simplify the algebra, you can assume the following simple bound: if $p(z) = a_d z^d + \cdots + a_0$ is of degree d, then there exists a positive real constant C such that for all $|z|$ sufficiently large we have $|p(z)| \geq C|z|^d$. (You are free to prove this result if you'd like to, but do not need to.)

(Recall that this is what we reduced the fundamental theorem of algebra to, so you have now proven the fundamental theorem of algebra!)

The above problem is directed towards Objectives 4 and 5 (harmonic functions and conformal mappings, and complex integration).

Survey. Estimate the amount of time you spent on each problem to the nearest half hour.

	Time Spent
Problem 1	
Problem 2	

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.