Homework 2

Complex analysis, lecture 4

Due September 15, 2025 by 11:59 PM

As usual, you may use any resources to solve these problems except where stated otherwise, with the exception of computational software/generative AI and posting these problems anywhere to be answered by others. Collaboration is encouraged, but everyone should write their own solutions. Write the names of any collaborators or sources used at the top of your homework. If you did not use any sources, write "sources used: none."

If you find any errors in either the homework or the lecture notes, please let me know, even if you are unsure whether it is an error or not.

As on most math problems, the mathematics is the issue, not the answer: whether you have a correct method is more important then whether you get to the correct number at the end, so include your method!

You do not have to simplify your answers completely (so for example $\frac{2}{2}$ is fine), but you do need to do all the computations (so for example if the problem is "find the largest value of $f(x)$," the answer "$f(3)$" is incomplete; you would also need to evaluate f at 3).

All problems in this homework are directed towards Objective 1 (complex numbers and algebra).

Problem 1. Consider the multivalued function $f(z) = \log(1 - z)$. Explain how to make a branch cut to define a single branch of f. In terms of this branch, what are the other branches? Describe what the resulting Riemann surface looks like.

The above problem is directed towards Objective 2 (branch cuts and Riemann surfaces).

Problem 2. Let α, β be complex numbers, and set $f(z) = z^\alpha(1 - z)^\beta$.

(a) What are the phase factors of f around $z_0 = 0$ and $z_0 = 1$?

(b) Consider a branch cut from 0 to 1. Under what conditions on α and β is f single-valued and continuous outside of this branch cut, i.e. on $\mathbb{C} \setminus [0, 1]$?

The above problem is directed towards Objective 2 (branch cuts and Riemann surfaces).

Problem 3. Let D be a domain, and write $\overline{D}$ for the domain $\{\overline{z} : z \in D\}$, the image of D under complex conjugation. Let f be an analytic function on D. Show that $z \mapsto \overline{f(\overline{z})}$ is an analytic function on $\overline{D}$.

The above problem is directed towards Objective 3 (analytic functions and the Cauchy–Riemann equations).

Problem 4. Show that if f and $\overline{f}$ are both analytic functions on a domain D, then f is constant on D. (Here $\overline{f}$ denotes the function sending z to $\overline{f(z)}$.)

The above problem is directed towards Objective 3 (analytic functions and the Cauchy–Riemann equations).

Problem 5. If $f = u + iv$ with u and v real-valued functions, we say that f is harmonic (as a complex-valued function) if u and v are both harmonic. In this language, we showed in class that if f is analytic, then it is harmonic.

Show that if $f(z)$ is harmonic and $zf(z)$ is also harmonic, then f is analytic. Give an example of a complex-valued function f which is harmonic but not analytic, showing that the second criterion is also necessary.

The above problem is directed towards Objectives 3 and 4 (analytic functions and the Cauchy–Riemann equations, and harmonic functions and conformal mappings).

Survey. Estimate the amount of time you spent on each problem to the nearest half hour.

	Time Spent
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.